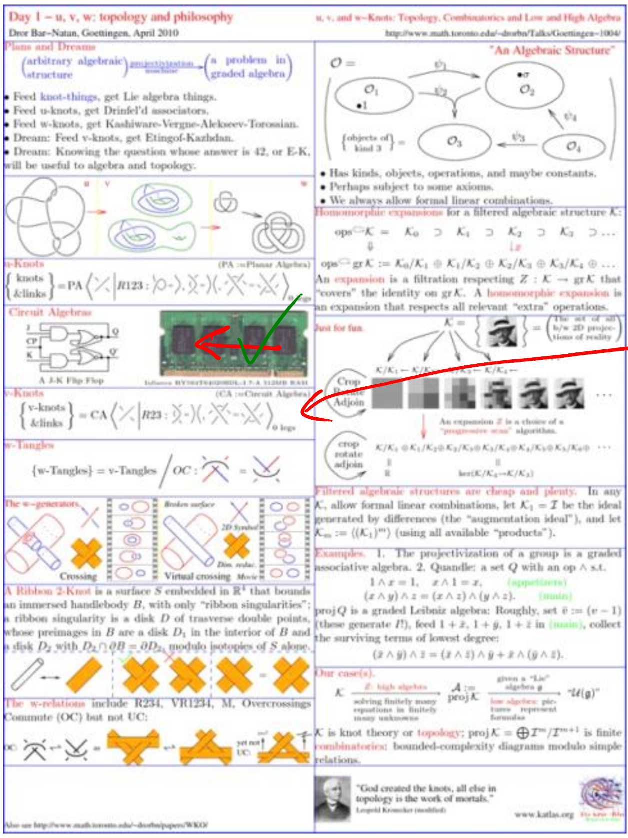
Goettingen - Monday preps

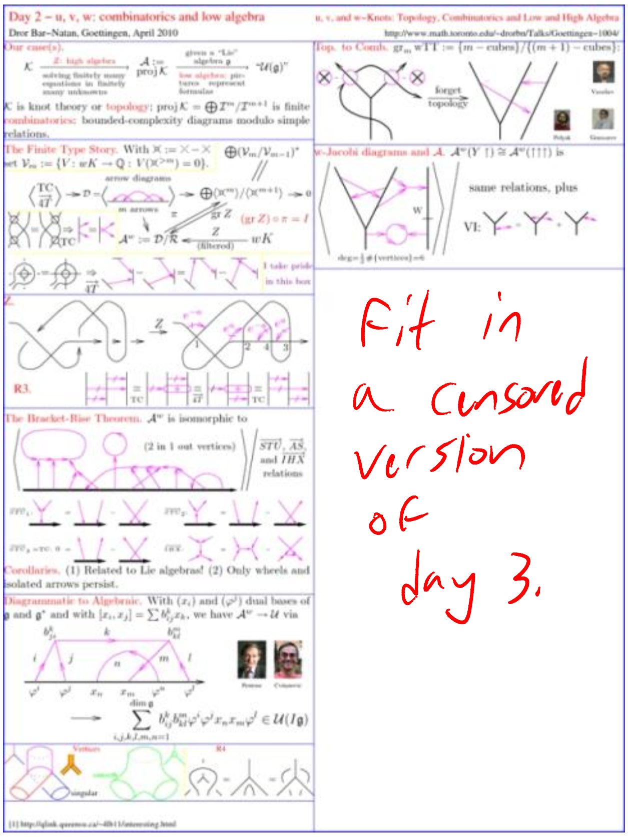
April-25-10
7:47 PM


Day 1 – u, v, w: topology and philosophy
Dror Bar-Natan, Goettingen, April 2010
u, v, and w-Knots: Topology, Combinatorics and Low and High Algebra
Plans and Dreams
• Feed knot-things, get Lie algebra things.
• Feed u-knots, get Drinfel'd associators.
• Feed w-knots, get Kashiware-Vergne-Alekseev-Torossian.
• Dream: Feed v-knots, get Etingof-Kazhdan.
• Dream: Knowing the question whose answer is 42, or E-K, will be useful to algebra and topology.
"An Algebraic Structure"
• Has kinds, objects, operations, and maybe constants.
• Perhaps subject to some axioms.
• We always allow formal linear combinations.
Homomorphic expansions for a filtered algebraic structure $\mathcal{K}$:
u-Knots
Circuit Algebras
A J-K Flip Flop
v-Knots
w-Tangles
The w-generators.
A Ribbon 2-Knot is a surface S embedded in $\mathbb{R}^4$ that bounds an immersed handlebody B, with only "ribbon singularities"; a ribbon singularity is a disk D of trasverse double points, whose preimages in B are a disk D_1 in the interior of B and a disk D_2 with $D_2 \cap \partial B = \partial D_2$, modulo isotopies of S alone.
The w-relations include R234, VR1234, M, Overcrossings Commute (OC) but not UC:
Filtered algebraic structures are cheap and plenty.
Examples. 1. The projectivization of a group is a graded associative algebra. 2. Quandle: a set Q with an op $\wedge$ s.t.
Our case(s).
$\mathcal{K}$ is knot theory or topology; proj $\mathcal{K}$ is finite combinatorics: bounded-complexity diagrams modulo simple relations.
"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)
www.katlas.org

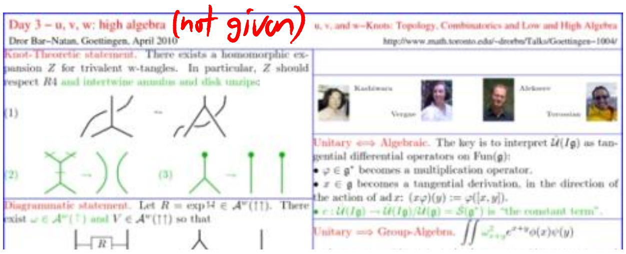
add PA.

* Partition!

* Fix logging in HBJs.

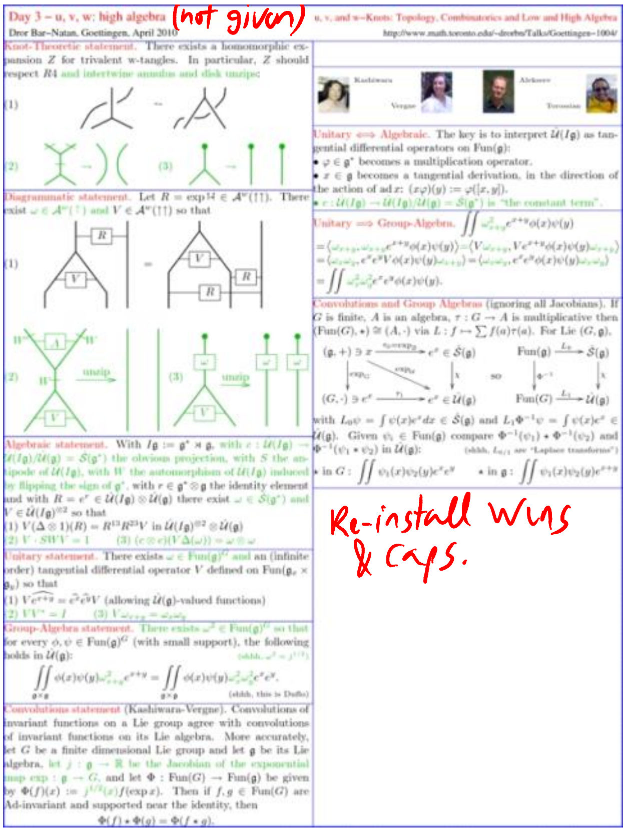

Day 2 – u, v, w: combinatorics and low algebra
Dror Bar-Natan, Goettingen, April 2010
u, v, and w-Knots: Topology, Combinatorics and Low and High Algebra
Our case(s).
Top. to Comb.
The Finite Type Story.
w-Jacobi diagrams and $\mathcal{A}$.
same relations, plus
The Bracket-Rise Theorem.
Corollaries. (1) Related to Lie algebras! (2) Only wheels and isolated arrows persist.
Diagrammatic to Algebraic.

Fit in a censored version of day 3.


Day 3 – u, v, w: high algebra
Dror Bar-Natan, Goettingen, April 2010
u, v, and w-Knots: Topology, Combinatorics and Low and High Algebra
Knot-Theoretic statement. There exists a homomorphic expansion Z for trivalent w-tangles.
Unitary ⟺ Algebraic.
Diagrammatic statement.
Unitary ⟹ Group-Algebra.

(not given)

* Re-insert "operations"



Day 3 – u, v, w: high algebra (not given)

Dror Bar-Natan, Goettingen, April 2010

u, v, and w–Knots: Topology, Combinatorics and Low and High Algebra

http://www.math.toronto.edu/~drorbn/Talks/Goettingen-1004/

Knot-Theoretic statement. There exists a homomorphic expansion Z for trivalent w-tangles. In particular, Z should respect $R4$ and intertwine annulus and disk unzips:

(1) (2) (3)

Diagrammatic statement. Let $R = \exp{}^{\uparrow\downarrow} \in \mathcal{A}^w(\uparrow\uparrow)$. There exist $\omega \in \mathcal{A}^w(\uparrow)$ and $V \in \mathcal{A}^w(\uparrow\uparrow)$ so that

(1) (2) unzip (3) unzip

Algebraic statement. With $I\mathfrak{g} := \mathfrak{g}^* \rtimes \mathfrak{g}$, with $c : \mathcal{U}(I\mathfrak{g}) \to \mathcal{U}(I\mathfrak{g})/\mathcal{U}(\mathfrak{g}) = \mathcal{S}(\mathfrak{g}^*)$ the obvious projection, with S the antipode of $\mathcal{U}(I\mathfrak{g})$, with W the automorphism of $\mathcal{U}(I\mathfrak{g})$ induced by flipping the sign of $\mathfrak{g}^*$, with $r \in \mathfrak{g}^* \otimes \mathfrak{g}$ the identity element and with $R = e^r \in \hat{\mathcal{U}}(I\mathfrak{g}) \otimes \hat{\mathcal{U}}(\mathfrak{g})$ there exist $\omega \in \hat{\mathcal{S}}(\mathfrak{g}^*)$ and $V \in \hat{\mathcal{U}}(I\mathfrak{g})^{\otimes 2}$ so that

(1) $V(\Delta \otimes 1)(R) = R^{13}R^{23}V$ in $\hat{\mathcal{U}}(I\mathfrak{g})^{\otimes 2} \otimes \hat{\mathcal{U}}(\mathfrak{g})$

(2) $V \cdot SWV = 1$ (3) $(c \otimes c)(V\Delta(\omega)) = \omega \otimes \omega$

Unitary statement. There exists $\omega \in \mathrm{Fun}(\mathfrak{g})^G$ and an (infinite order) tangential differential operator V defined on $\mathrm{Fun}(\mathfrak{g}_x \times \mathfrak{g}_y)$ so that

(1) $V\widehat{e^{x+y}} = \hat{e^x}\hat{e^y}V$ (allowing $\hat{\mathcal{U}}(\mathfrak{g})$-valued functions)

(2) $VV^* = I$ (3) $V\omega_{x+y} = \omega_x\omega_y$

Group-Algebra statement. There exists $\omega^2 \in \mathrm{Fun}(\mathfrak{g})^G$ so that for every $\phi, \psi \in \mathrm{Fun}(\mathfrak{g})^G$ (with small support), the following holds in $\hat{\mathcal{U}}(\mathfrak{g})$: (shhh, $\omega^2 = j^{1/2}$)

$$\iint_{\mathfrak{g}\times\mathfrak{g}} \phi(x)\psi(y)\omega^2_{x+y}e^{x+y} = \iint_{\mathfrak{g}\times\mathfrak{g}} \phi(x)\psi(y)\omega^2_x\omega^2_y e^x e^y.$$

(shhh, this is Duflo)

Convolutions statement (Kashiwara-Vergne). Convolutions of invariant functions on a Lie group agree with convolutions of invariant functions on its Lie algebra. More accurately, let G be a finite dimensional Lie group and let $\mathfrak{g}$ be its Lie algebra, let $j : \mathfrak{g} \to \mathbb{R}$ be the Jacobian of the exponential map $\exp : \mathfrak{g} \to G$, and let $\Phi : \mathrm{Fun}(G) \to \mathrm{Fun}(\mathfrak{g})$ be given by $\Phi(f)(x) := j^{1/2}(x)f(\exp x)$. Then if $f, g \in \mathrm{Fun}(G)$ are Ad-invariant and supported near the identity, then

$$\Phi(f) \star \Phi(g) = \Phi(f \star g).$$

Kashiwara Vergne Alekseev Torossian

Unitary $\Longleftrightarrow$ Algebraic. The key is to interpret $\hat{\mathcal{U}}(I\mathfrak{g})$ as tangential differential operators on $\mathrm{Fun}(\mathfrak{g})$:

- $\varphi \in \mathfrak{g}^*$ becomes a multiplication operator.
- $x \in \mathfrak{g}$ becomes a tangential derivation, in the direction of the action of $\mathrm{ad}\, x$: $(x\varphi)(y) := \varphi([x, y])$.
- $c : \mathcal{U}(I\mathfrak{g}) \to \mathcal{U}(I\mathfrak{g})/\mathcal{U}(\mathfrak{g}) = \mathcal{S}(\mathfrak{g}^*)$ is "the constant term".

Unitary $\Longrightarrow$ Group-Algebra. $\iint \omega^2_{x+y}e^{x+y}\phi(x)\psi(y)$

$= \langle \omega_{x+y}, \omega_{x+y}e^{x+y}\phi(x)\psi(y)\rangle = \langle V\omega_{x+y}, Ve^{x+y}\phi(x)\psi(y)\omega_{x+y}\rangle$
$= \langle \omega_x\omega_y, e^xe^yV\phi(x)\psi(y)\omega_{x+y}\rangle = \langle \omega_x\omega_y, e^xe^y\phi(x)\psi(y)\omega_x\omega_y\rangle$
$= \iint \omega^2_x\omega^2_y e^xe^y\phi(x)\psi(y).$

Convolutions and Group Algebras (ignoring all Jacobians). If G is finite, A is an algebra, $\tau : G \to A$ is multiplicative then $(\mathrm{Fun}(G), \star) \cong (A, \cdot)$ via $L : f \mapsto \sum f(a)\tau(a)$. For Lie $(G, \mathfrak{g})$,

$$\begin{array}{ccc} (\mathfrak{g}, +) \ni x & \xrightarrow{\tau_0 = \exp_S} & e^x \in \hat{\mathcal{S}}(\mathfrak{g}) \\ \downarrow \exp_G & \searrow \exp_U & \downarrow \chi \\ (G, \cdot) \ni e^x & \xrightarrow{\tau_1} & e^x \in \hat{\mathcal{U}}(\mathfrak{g}) \end{array} \quad \text{so} \quad \begin{array}{ccc} \mathrm{Fun}(\mathfrak{g}) & \xrightarrow{L_0} & \hat{\mathcal{S}}(\mathfrak{g}) \\ \downarrow \Phi^{-1} & & \downarrow \chi \\ \mathrm{Fun}(G) & \xrightarrow{L_1} & \hat{\mathcal{U}}(\mathfrak{g}) \end{array}$$

with $L_0\psi = \int \psi(x)e^x dx \in \hat{\mathcal{S}}(\mathfrak{g})$ and $L_1\Phi^{-1}\psi = \int \psi(x)e^x \in \hat{\mathcal{U}}(\mathfrak{g})$. Given $\psi_i \in \mathrm{Fun}(\mathfrak{g})$ compare $\Phi^{-1}(\psi_1) \star \Phi^{-1}(\psi_2)$ and $\Phi^{-1}(\psi_1 \star \psi_2)$ in $\hat{\mathcal{U}}(\mathfrak{g})$: (shhh, $L_{0/1}$ are "Laplace transforms")

- in G: $\iint \psi_1(x)\psi_2(y)e^xe^y$ • in $\mathfrak{g}$: $\iint \psi_1(x)\psi_2(y)e^{x+y}$

Re-install wuns & caps.

* Re-insert "operations"

Go 18C 2?!

Fix it?

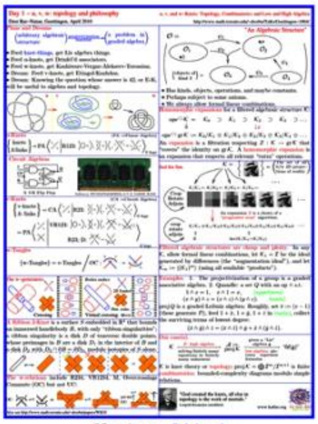
u, v, and w-Knots: Topology, Combinatorics and Low and High Algebra

Courant Lecture Series
Goettingen April 27,28,29, 2010

Overall Abstract. I will discuss three types of knotted objects - the "u" type, for "usual", the "v" type, for "virtual", and the "w" type, for "welded", or "weakly virtual", or "warm up". I will then discuss an abstract and general yet rather simple machine that in a uniform manner associates to each such class of knotted objects a "combinatorics", and a "low algebra", and a "high algebra". The latter is high indeed - it is the theory of Drinfel'd associators in the u case, most likely it is the Etingof-Kazhdan theory of quantization of Lie bi-algebras in the v case, and it is the Kashiwara-Vergne theory of convolutions on Lie groups and algebras in the w case. Thus these three pieces of high algebra have a simple topological origin. And as on the level of topology u, v, and w are tied together, their respective high algebra theories are closely related, with some of these relationships clearly understood, and some that are yet to be explored.

Day 1
u, v, w: topology and philosophy

- Dreams and plans.
- Knots, planar diagrams, Reidemeister moves, virtual knots are to knots as manifolds are to Euclidean spaces, flying rings and knotted tubes in 4D and w-knots.
- Planar algebras and circuit algebras.
- The abstract machine - filtered and graded spaces, expansions and homomorphic expansions, equations in graded spaces.



Handout - G1.html

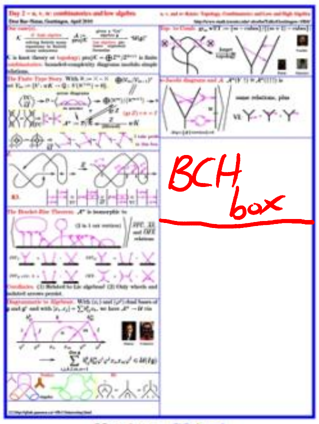
DBN A link to a video of this talk will be posted here.

Day 2
u, v, w: combinatorics and low algebra

- Finite type invariants, weight systems, chord diagrams, arrow diagrams, 4T relations
- The "bracket-rise" theorem, STU and IHX relations
- Maps into various kinds of universal enveloping algebras.

* The old "Day 3" with link

2 X **day handout** - G.pdf

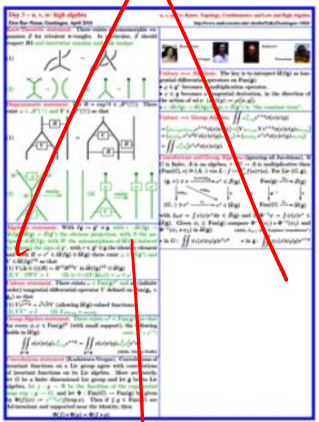


Handout - G2.html

DBN A link to a video of this talk will be posted here.

~~Day 3~~
~~**u, v, w: high algebra**~~

- ~~Kashiwara-Vergne and Alekseev-Torossian: convolutions, integrals, measure preserving transformations, unitary operators, universal formulas and universal equations.~~
- ~~A word on knotted trivalent graphs, Drinfel'd associators, and Chern-Simons-Witten theory.~~
- ~~Dreams on v-knots, Etingof-Kazhdan, and quantization of Lie bi-algebras.~~
- ~~Hallucinations on knot homologies and on further physics.~~



Handout - G3.html

DBN A link to a video of this talk will be posted here.

All sources are in G.zip.

In Luming - place link to Goettingen.

New Day 3:
v: 18 conjectures, as not given in Luming
Luming link.
Back side - Z2A handout.

BCH box:

Theorem: A homomorphic expansion for wTT is the same as a solution $F\in\hat{\mathcal{U}}(tder_2)$ of BCH-like equation,

$$Fe^{x+y}=e^xe^y$$

equation, $$F \, e^{x+y} = e^x e^y$$