AS for Proj Q

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ef:quandle **Definition 6.5.** A quandle is a set Q with a binary operation $\uparrow : Q \times Q \to Q$ satisfying the following axioms:

(1) $\forall x \in Q$, $x \uparrow x = x$.
(2) For any fixed $y \in Q$, the map $x \mapsto x \uparrow y$ is invertible[29].
(3) Self-distributivity: $\forall x, y, x \in Q$, $(x \uparrow y) \uparrow z = (x \uparrow z) \uparrow (y \uparrow z)$.

We say that a quandle Q has a unit, or is unital, if there is a distiguished element $1 \in Q$ satisfying the further axiom:

(4) $\forall x \in Q$, $x \uparrow 1 = x$ and $1 \uparrow x = 1$.

If G is a group, it is also a (unital) quandle by setting $x \uparrow y := y^{-1}xy$, yet there are many quandles that do not arise from groups in this way.

prop:ProjQ **Proposition 6.6.** *If Q is a unital quandle,* $\mathrm{proj}_0 Q$ *is one-dimensional and* $\mathrm{proj}_{>0} Q$ *is a graded Lie algebra generated by* $\mathrm{proj}_1 Q$.

Proof. For any algebraic structure A with just one kind of objects, $\mathrm{proj}_0 A$ is one-dimensional, generated by the equivalence class $[x]$ of any single object x. In particular, $\mathrm{proj}_0 Q$ is one-dimensional and generated by $[1]$. Let $\mathcal{I} \subset \mathbb{Q}Q$ be the augmentation ideal of Q. For any $x \in Q$ set $\bar{x} := x - 1 \in \mathcal{I}$. Then $\mathcal{I}$ is generated by the $\bar{x}$'s, and therefore $\mathcal{I}^m$ is generated by expressions involving the operation $\uparrow$ applied to some m elements of $\bar{Q} := \{\bar{x} : x \in Q\}$ and possibly some further elements $y_i \in Q$. When regarded in $\mathcal{I}^m/\mathcal{I}^{m+1}$, any y_i is such a generating expression can be replaced by 1, for the difference would be the same expression with y_i replaced by $\bar{y}_i$, and this is now a member of $\mathcal{I}^{m+1}$. But for any element $z \in \mathcal{I}$ we have $z \uparrow 1 = z$ and $1 \uparrow z = 0$, so all the 1's can be eliminated from the expressions generating $\mathcal{I}^m$. Thus $\mathrm{proj}_{>0} Q$ is generated by $\bar{Q}$ and hence by $\mathrm{proj}_1 Q$.

Let $\Delta : \mathbb{Q}Q \to \mathbb{Q}Q \otimes \mathbb{Q}Q$ be the linear extension of the operation $x \mapsto x \otimes x$ defined on $x \in Q$, and extend $\uparrow$ to a binary operator $\uparrow_2 : (\mathbb{Q}Q \otimes \mathbb{Q}Q) \otimes (\mathbb{Q}Q \otimes \mathbb{Q}Q) \to \mathbb{Q}Q \otimes \mathbb{Q}Q$ by using $\uparrow$ twice, to pair the first and third tensor factors and then to pair the second and the fourth tensor factors. With this language in place, the self-distributivity axiom becomes the following *linear* statement, which holds for every $x, y, z \in \mathbb{Q}Q$:

$$(x \uparrow y) \uparrow z = \uparrow \circ \uparrow_2 (x \otimes y \otimes \Delta z). \qquad (25)$$ eq:LinSelf

Clearly, we need to understand Δ better. By direct computation, if $x \in Q$ then $\Delta\bar{x} = \bar{x} \otimes 1 + 1 \otimes \bar{x} + \bar{x} \otimes \bar{x}$. We claim that in general, if z is a generating expression of $\mathcal{I}^m$ (that is, a formula made of m elements of $\bar{Q}$ and $m-1$ applications of $\uparrow$), then

$$\Delta z = z \otimes 1 + 1 \otimes z + \sum z'_i \otimes z''_i, \qquad \text{with} \qquad \sum z'_i \otimes z''_i \in \sum_{\substack{m'+m''=m+1,\\ m',m''>0}} \mathcal{I}^{m'} \otimes \mathcal{I}^{m''}. \qquad (26)$$ eq:Quandle

Indeed, for the generators of $\mathcal{I}^1$ this had just been shown, and if $z = z_1 \uparrow z_2$ is a generator of $\mathcal{I}^m$, with z_1 and z_2 generators of $\mathcal{I}^{m_1}$ and $\mathcal{I}^{m_2}$ with $1 \le m_1, m_2 < m$ and $m_1 + m_2 = m$,

foot:upinv [29] This can alternatively be stated as "there exists a second binary operation $\uparrow^{-1}$ so that $\forall x$, $x = (x \uparrow y) \uparrow^{-1} y = (x \uparrow^{-1} y) \uparrow y$", so this axiom can still be phrased within the language of "algebraic structures". Yet note that below we do not use this axiom at all.

then (using $w\uparrow 1 = w$ and $1\uparrow w = 0$ for $w \in \mathcal{I}$),

$$\begin{aligned}
\Delta z &= \Delta(z_1\uparrow z_2) = (\Delta z_1)\uparrow_2(\Delta z_2) \\
&= (z_1 \otimes 1 + 1 \otimes z_1 + \sum z'_{1j} \otimes z''_{1j})\uparrow_2(z_2 \otimes 1 + 1 \otimes z_2 + \sum z'_{2k} \otimes z''_{2k}) \\
&= (z_1\uparrow z_2) \otimes 1 + 1 \otimes (z_1\uparrow z_2) \\
&\quad + \sum_j \left((z'_{1j}\uparrow z_2) \otimes z''_{1j} + z'_{1j} \otimes (z''_{1j}\uparrow z_2) + \sum_k (z'_{1j}\uparrow z'_{2k}) \otimes (z''_{1j}\uparrow z''_{2k}) \right),
\end{aligned}$$

and it is easy to see that the last line agrees with (26).

We can now combine Equations (25) and (26) to get that for any $x, y, z \in \mathbb{Q}Q$,

$$(x\uparrow y)\uparrow z = (x\uparrow z)\uparrow y + x\uparrow(y\uparrow z) + \sum (x\uparrow z'_i)\uparrow(y\uparrow z''_i).$$

If $x \in \mathcal{I}^{m_1}$, $y \in \mathcal{I}^{m_2}$, and $z \in \mathcal{I}^{m_3}$, then by (26) the last term above is in $\mathcal{I}^{m_1+m_2+m_3+1}$, and so the above identity becomes the Jacobi identity $(x\uparrow y)\uparrow z = (x\uparrow z)\uparrow y + x\uparrow(y\uparrow z)$ in $\mathrm{proj}_{m_1+m_2+m_3} Q$.

MORE. It remains to show that within $\mathrm{proj}_{>0} Q$, the operation $\uparrow$ is anti-symmetric.

Exercise 6.7. Verify that in the above proof axiom (2) of Definition 6.5 was not used. Verify also that if this axiom is introduced as in footnote 29 using a second operation $\uparrow^{-1}$ (thus enlarging the set of algebraic expressions that we need to consider as in MORE), Proposition6.6 remains true.

Compare with Leibnitz:

$$[x,[y,z]] = [[x,y],z] - [[x,z],y] \xrightarrow{\text{Flip } [\ ],}$$

$$[[z,y],x] = [z,[y,x]] - [y,[z,x]] \xrightarrow{(x,y,z)\mapsto(z,y,x)}$$

$$[[x,y],z] = [x,[y,z]] - [y,[x,z]]$$

$\Longrightarrow$ It would be lovely to know that

$$(x\uparrow z)\uparrow y = -\, y\uparrow(x\uparrow z)$$