Khovanskii@Colloq: Bernstein-Kushnirenko theorem and its generalizations

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Convex geometry: $A, B \subset \mathbb{R}^n \longrightarrow A+B = \{a+b : \begin{smallmatrix} a\in A \\ b \in B\end{smallmatrix}\}$

this is commutative and associative, but has no cancellation property: $A+B = A+C \not\Rightarrow B = C$

(Example: $A = C = [0,1]$, $B = \{0,1\}$)

Consider the Grothendieck group . . .

<u>claim</u> $B \sim C$ iff $\Delta(B) = \Delta(C)$ ← convex hull

So we may as well use only convex sets.

A function $f : \mathcal{L} \to \mathbb{R}$, where $\mathcal{L}$ is an infinite dimensional space, is a polynomial of degree n if it is of the form $f(x) = B(x, x \ldots, x)$ where B is multi-linear $\mathcal{L}^{\otimes n} \to \mathbb{R}$.

<u>Thm</u> (Minkowski) The obvious extension of vol to the convex-set-group in $\mathbb{R}^n$ is polynomial of deg n.

---- Can define "mixed volumes" $V(\Delta_1 \ldots \Delta_n)$, using the multilinear form corresponding to Vol.

Notation: $\mathbb{X} = x_1 \ldots x_n \quad \mathbb{X}^{\mathbb{k}} = x_1^{k_1} \cdots x_n^{k_n}$

P: A "Laurent poly" $\sum a_{\mathbb{k}} \mathbb{X}^{\mathbb{k}}$

$\Delta(P) =$ Newton poly of P = Convex hull of non-zero coeffs

$P : (\mathbb{C}^*)^n \to \mathbb{C}$ Consider the system

$P_1 = \ldots = P_n = 0$ w/ P_i Laurent, how many solutions are there in $(\mathbb{C}^*)^n$? $\Delta(f_i)=\Delta_i$

Bernstein-Kushnirenko (1975): The number of sol'ns is

$n!\,V(\Delta_1 \dots \Delta_n)$ (generically)

Properties of mixed volumes:

1. monotonicity: $\Delta_i \supseteq \Delta_i' \Rightarrow V(\Delta_i) \geq V(\Delta_i')$

The K-Kaveh generalization:

$(\mathbb{C}^*)^n \longrightarrow X$

$A \subset \mathbb{Z}^n \Leftrightarrow \{p = \sum_{k \in A} a_k x^k\} = L_A$

given $L_{A_1} \dots L_{A_n}$, what's $|\{\forall i\ f_i = 0\}|$ ($f_1 \in L_{A_1}, \dots, f_2 \in L_{A_n}$)

$L_1 \dots L_n \subset \mathbb{C}(X)$

f.d. subspaces of the space of all rational fncns on X

$B(L_1, \dots L_n) := |\{f_1 = \dots = f_n = 0\}|$, w/ generic $f_i \in L_i$

in count, ignore points in which one of the L_i is uniformly 0, and in which there's a pole.

$L = L_1 \cdot L_2 := \mathrm{Span}\{\text{all products}\}$

Claim B is "multi-linear" using $L_1 L_2$ replacing addition

K-K: B has many properties analog to the props of mixed volume.

Also, there is some assignment on "Newton Polyhedra" $\Delta(L) \subset \mathbb{R}^n$ to L's as above, w/ an analogue of the B-K theorem.

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