Guth@GSS: Inequalities about lengths, areas and volumes

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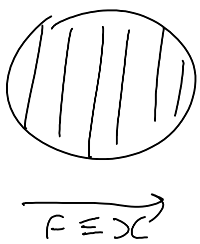
Thm (Guth, 2004) If $U \subset \mathbb{R}^2$ is open bounded and ∂U is smooth and $\text{Area}(U)=1$, then $\exists F: U \to \mathbb{R}$ s.t. $\forall t$ $\text{length}(F^{-1}(t)) \leq 5$



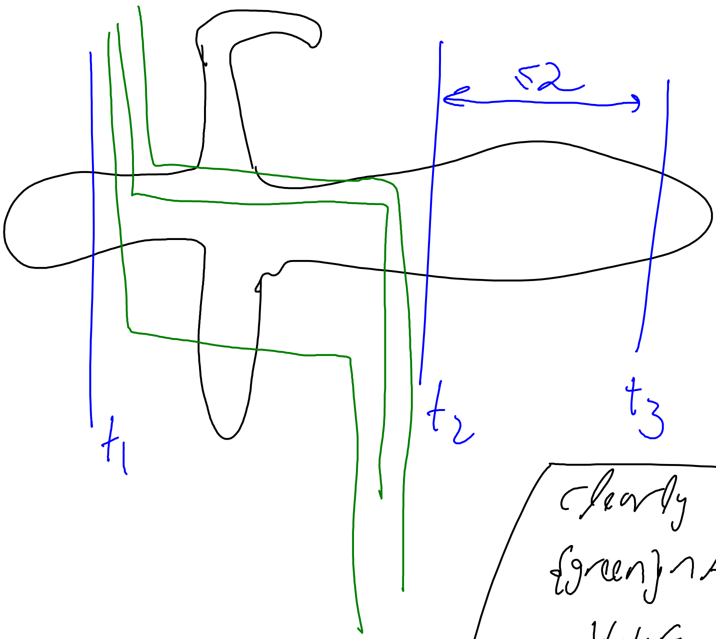
Def $\text{Width}(U) := \min_F \max_t \text{length}(F^{-1}(t))$

Thm is $\Leftrightarrow$ $\text{Width}(U)^2 \leq C \cdot \text{Area}(U)$

Pf of Thm

$$1 = A(U) = \int_{-\infty}^{\infty} \text{length}(U \cap \{x=t\})\, dt$$

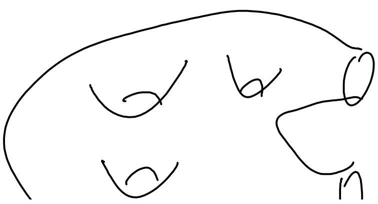
So $\forall n$ $\exists t_n \in [n, n+1]$ s.t. $\text{length}(U \cap [x=t_n]) < 1$



In between the blues, build the level sets of F to be the green zig zag curves as on right.

Clearly the length of every $\{\text{green}\} \cap A$ is less than $4+\epsilon$.

Curved surfaces: $\Sigma^2 \subset \mathbb{R}^3$



The isoperimetric ineq:

$\text{Area}(\Sigma) \leq \text{len}(\partial\Sigma)^2$

$\mathrm{Area}(C) \leq \mathrm{Vol}(\partial \Sigma)^2$

is clearly false. Also

$$\mathrm{Width}(\Sigma)^2 \leq C \,\mathrm{Area}(\Sigma) \quad \text{is false}$$

Example: thicken the 1-skeleton of the cubical grid just a bit, and look at the boundary surface. Yet

Thm If Σ^2 is a disk then

$$\mathrm{width}(\Sigma)^2 \leq 10^4 \,\mathrm{Area}(\Sigma)$$

Higher Dimensions: $U^3 \subset \mathbb{R}^3$

$$\mathrm{width}(U) := \inf_{F: U \to \mathbb{R}} \sup_{t \in \mathbb{R}} \left(\mathrm{Area}(F^{-1}(t)) \right)$$

The width-area thm generalizes, but no such Thms for $M^3 \subset \mathbb{R}^4$.

Loewner's thm (1940s) If Σ^2 is a curved T^2 then $\exists \gamma \subset \Sigma^2$ top. non-trivial s.t.

$$\mathrm{Length}(\gamma)^2 \leq 10 \cdot \mathrm{Area}(\Sigma)$$

In higher dimension:

Gromov (1983): If M^n is a curved T^n then there is a $\gamma \subset M^n$ top. non-trivial with

$$\mathrm{length}(\gamma)^n \leq C \cdot \mathrm{Vol}(M)$$