BEER-1 by Peter Lee, October 8

October-08-09
11:14 AM

BEER is ...

Title: Groups and Lie algebras corresponding to the Yang-Baxter equations

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Abstract: For a positive integer n we introduce quadratic Lie algebras tr_n qtr_n and discrete groups Tr_n, QTr_n naturally associated with the classical and quantum Yang-Baxter equation, respectively. We prove that the universal enveloping algebras of the Lie algebras tr_n, qtr_n are Koszul, and find their Hilbert series. We also compute the cohomology rings of these Lie algebras (which by Koszulity are the quadratic duals of the enveloping algebras). We construct cell complexes which are classifying spaces of the groups Tr_n and QTr_n, and show that the boundary maps in them are zero, which allows us to compute the integral cohomology of these groups. We show that the Lie algebras tr_n, qtr_n map onto the associated graded algebras of the Malcev Lie algebras of the groups Tr_n, QTr_n, respectively. In the case of Tr_n, we use quantization theory of Lie bialgebras to show that this map is actually an isomorphism. At the same time, we show that the groups Tr_n and QTr_n are not formal for n>3.

"descending vB_n"

Objective: Find a "classifying space" for vB_n, dvB_n,

(we'll find $dvB_n \hookrightarrow vB_n \twoheadrightarrow dvB_n$)

and compute homology.

Recall A BG for a group G is a top. space which is path connected, $\pi_1(BG)=G$, $\pi_{n\geq 2}(BG)=$ trivial.

Equivalently, $\exists\ EG \xrightarrow{\pi} BG$ s.t. EG is contractible and π is a covering map w/ fiber G.

Strategy For dvB_n we construct a complex P_n (combinatorially, but realizable in $\mathbb{R}^n$), with $BdvB_n$ a quotient C_n of P_n by a certain action.

The Permutohedron P_n: Let P_n be the poset of ordered partitions of $[n]=\{1,\dots n\}$, "poset" by "refinement".

ordered partition $:= [n] = S_1 \sqcup S_2 \sqcup S_3$ $\quad (\neq S_2 \sqcup S_1 \sqcup S_3)$

with each S_i non-empty and un-ordered.

P_n: $[n]$ is smallest, bigger things are "faces"

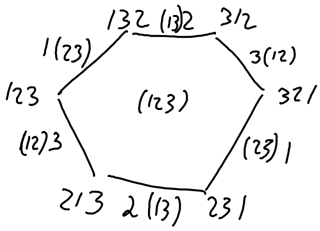
$[n]$ is [illegible] any partitioning reduces th[illegible]

$[1]$ is $n-1$ dimensional, with partitioning ... the dimensions, permutations are points.

<u>Examples</u>

P_2: $(1)(2) \xrightarrow{\;(12)\;} (2)(1)$

P_3



Geometric Realization: Let A_n be the hyperplane in $\mathbb{R}^n$ defined by $\sum_{i=1}^{n} x_i = 1+2+\dots+n = \frac{n(n+1)}{2}$

Consider the point $(n, n-1, \dots 1) = x$; it an its orbit under S_n are on A_n. Let

$$P_n := \mathrm{Conv}(S_n x)$$

Faces: $R_1 \sqcup R_2 \sqcup \dots \sqcup R_r =$

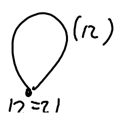
$$\Big\{(x_1, \dots x_n) : \forall 1 \le i \le r \ \sum_{s \in R_i} x_s = (|R_1| + \dots + |R_{i-1}| + 1) + \dots + (|R_1| + \dots + |R_i|)\Big\}$$

<u>S_r Action</u> Since the $(n-r)$-dim faces $\leftrightarrow \{T_1 \sqcup \dots \sqcup T_r\}$, there is an action of S_r on the $(n-r)$-dim faces.

Suppose $R_1 \sqcup \dots \sqcup R_r$ is a face of $T_1 \sqcup \dots \sqcup T_s$, then there is a surjection $f: [r] \to [s]$ which is order preserving. .... See BBS.

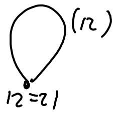
We mod out by all S_r actions: $C_n := P_n / \substack{\text{all } S_r \text{ actions} \\ \text{for } 2 \le r \le n}$

C_2:



C_3:

$\cup$ a 2-cell.

C_2 : 

C_3 : 

$\cup$ a 2-cell.

Thm C_n is a classifying space for dvB_n.

Pf of $\pi_1(C_n) = dvB_n$: π_1 is generated by simple loops $\{a,b\} := (ab)$-singletons-- $\longleftrightarrow \sigma_{ab} \in dvB_n$

and the faces lead to the R3 relations
(and localities)