Kac Moody Algebras

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Some wikipedia wisdom

A Kac–Moody algebra is given by the following:

1. An n by n generalized Cartan matrix $C = (c_{ij})$ of rank r.
2. A vector space $\mathfrak{h}$ over the complex numbers of dimension $2n - r$.
3. A set of n linearly independent elements α_i of $\mathfrak{h}$ and a set of n linearly independent elements α_i^* of the dual space, such that $\alpha_i^*(\alpha_j) = c_{ij}$. The α_i are known as **coroots**, while the α_i^* are known as **roots**.

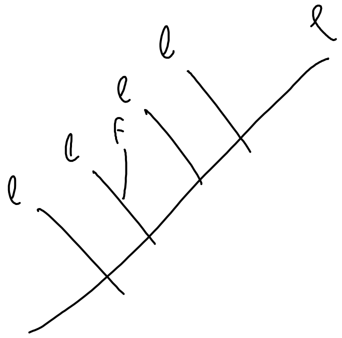
The Kac–Moody algebra is the Lie algebra $\mathfrak{g}$ defined by generators e_i and f_i and the elements of $\mathfrak{h}$ and relations

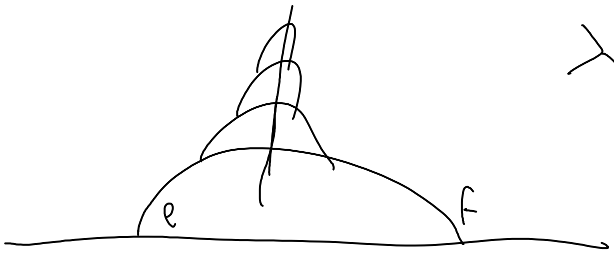
- $[e_i, f_i] = \alpha_i$
- $[e_i, f_j] = 0$ for $i \neq j$
- $[e_i, x] = \alpha_i^*(x)e_i$, for $x \in \mathfrak{h}$
- $[f_i, x] = -\alpha_i^*(x)f_i$, for $x \in \mathfrak{h}$
- $[x, x'] = 0$ for $x, x' \in \mathfrak{h}$
- $\mathrm{ad}(e_i)^{1-c_{ij}}(e_j) = 0$
- $\mathrm{ad}(f_i)^{1-c_{ij}}(f_j) = 0$

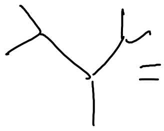
A **generalized Cartan matrix** is a square matrix $A = (a_{ij})$ with integer entries such that

1. For diagonal entries, $a_{ii} = 2$.
2. For non-diagonal entries, $a_{ij} \leq 0$.
3. $a_{ij} = 0$ if and only if $a_{ji} = 0$
4. A can be written as DS, where D is a diagonal matrix, and S is a symmetric matrix.

The third condition is not independent but is really a consequence of the first and fourth conditions.







$$[[e_i, e_j], [f_i, f_j]] = [[[e_i, e_j], f_i], f_j] + [f_i, [[e_i, e_j], f_j]]$$

$$= [[\alpha_i, e_j], f_j] + [f_i, [e_i, \alpha_j]]$$

$$= [-\alpha_j^*(\alpha_i) e_j, f_j] + [f_i, \alpha_i^*(\alpha_j) e_i]$$

$$= -\alpha_j^*(\alpha_i)\alpha_j - \alpha_i^*(\alpha_j)\alpha_i$$

$$= -c_{ji}\alpha_j - c_{ij}\alpha_i$$