Beck: Quantum algebras and quiver modules

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4:10 PM

$\mathfrak{g}$ – Kac-Moody Lie algebra

generalized Cartan Matrix

$$\left.\begin{array}{l} a_{ii}=2, \quad a_{ij}\le 0 \quad \text{if } i\ne j \\ a_{ij}=0 \iff a_{ji}=0 \end{array}\right\} A$$

If A is pos. definite, $\mathfrak{g}$ is simple

Pos semi-def –"– affine

otherwise –"– is wild.

$$\mathfrak{g} = \langle e_i, f_i, h_i\rangle / \mathrm{ad}^{1-a_{ij}}(x) = *$$

root system $= \Delta \supset \Delta^+ \supset \Pi = \{\alpha_1 \dots \alpha_n\}$

$\mathfrak{g} \supset \mathfrak{n}^+$ nilpotent subalgebra

$W = \langle s_i \mid$ simple reflections$\rangle$

$U^+ := U(\mathfrak{n}^+)$

Consider $\mathbb{Z}[q, q^{-1}]$, set $[n] = \dfrac{q^n - q^{-n}}{q - q^{-1}}$

$[n]! = [n][n-1]\dots \qquad [0]! = 1$

associative algebra w/ these generators

Def $U_q^+ = \langle E_i\rangle / \mathrm{ad}_q^{1-a_{ij}}(E_i)E_j = 0$

where e.g.,

$$\left.\mathrm{ad}_q^2(A)(B) = A^2B - [2]ABA + BA^2\right\} \text{abstract formulation?}$$

Fact $\forall w \in W \;\exists\, T_w \in \mathrm{aut}(U_q)$

Using T_w we define a basis of U_q^+:
Let $w_0 \in W$ be the longest element in the Weyl group.

assume $\mathfrak{g}$ is simple &

$w_0 = s_{i_N} s_{i_{N-1}} \ldots s_{i_1}$ a product of simple reflections of minimal length.

$$\alpha_{i_N} < s_{i_N}\alpha_{i_{N-1}} < s_{i_N}s_{i_{N-1}}\alpha_{i_{N-2}} \quad \ldots$$
$$\beta_N \qquad \beta_{N-1} \quad \ldots$$

$\Delta^+ = \{\beta_i\}$

set $E_{\beta_j} = T_{i_N} T_{i_{N-1}} \ldots T_{i_{j+1}} E_{i_j}$

For $c \in \mathbb{Z}_{\geq 0}^N$ set

$$L(c) = E_{\beta(N)}^{c(N)} \ldots E_{\beta_1}^{c(1)}$$ "the PBW basis"

<u>Thm</u> (Lusztig) For each $c \in \mathbb{Z}_{\geq 0}^N$ there exists a unique $b(c) \in U_q^+$ s.t.

1. $\overline{b(c)} = b(c)$ $\qquad \bar{\ } : U_q^+ \circlearrowleft$ by $q \to q^{-1}$
2. $b(c) = L(c) + \sum_{\substack{c' < c \\ \text{in lex}}} a_{cc'} L(c')$

$b(c)$ are "the canonical basis of U_q^+". 0:22