Pensieve Header: An improved MVA program with Jana Archibald.

```
<< KnotTheory`
```

```
Loading KnotTheory` version of January 20, 2009, 17:44:36.984.
Read more at http://katlas.org/wiki/KnotTheory.
```

```
MultivariableAlexander2[PD[Loop[_]]] := 1 &
MultivariableAlexander2[K_] /; Head[K] =!= PD := MultivariableAlexander2[PD[K]]

MultivariableAlexander2[pd_PD] := MultivariableAlexander2[pd] = Module[
  {l, mat, skel, pd1, G, t, c, path, i, j, k, M, emb, done, pd2, rot, place},
  l = Length[pd];
  mat = Table[0, {2 * l}, {2 * l}];
  skel = Skeleton[pd];
  pd1 = List @@ pd;
  G = Table[0, {2 * l}, {l}];
  pd1 //. X[a_, b_, c_, d_] :> If[d == b + 1 || b - d > 1,
     {mat[[c, a]] = -t[b]; mat[[c, b]] = t[a] - 1;  mat[[c, c]] = 1},
     {mat[[c, a]] = -1; mat[[c, b]] = 1 - t[a] ; mat[[c, c]] = t[b]}
    ];
  c = Times @@ pd /. {
      X[i_, j_, k_, l_] /; (l - j == 1 || j - l > 1) :> path[k] path[i] path[j, l],
      X[i_, j_, k_, l_] /; (j - l == 1 || l - j > 1) :> path[k] path[i] path[l, j],
      P[i_, j_] :> path[i, j]
     } //. {
     path[a__, i_] path[i_, b__] :> path[a, i, b],
     path[a__, i_] path[b__, i_] :> Join[path[a, i], Reverse[path[b]]],
     path[i_, a__] path[i_, b__] :> Join[Reverse[path[b]], path[i, a]],
     path[a__, i_] path[i_] :> path[a, i],
     path[i_, a__] path[i_] :> path[a, i],
     path[i_] path[i_] :> path[i]
    };
  For[i = 1, i ≤ 2 * l, i++,
   G = ReplacePart[G, 1, {i, First[First[Position[c, i]]]}]
  ];
  mat = mat /. t[a_] :> t[Position[skel, a][[1, 1]]];
  M = Factor[Simplify[
     Det[
       Delete[
        Transpose[Delete[
          Transpose[G].mat.G,
          Position[c, pd1[[1, 1]]][[1, 1]]
         ]],
        Position[c, pd1[[1, 1]]][[1, 1]]
```

```
          ]
        ] / (t[Position[skel, pd1[[1, 1]]][[1, 1]]] - 1)
      ]];
    emb = Table[Null, {Length[pd]}];
    done = Table[Null, {2 * Length[pd]}];
    emb[[1]] = 0;
    pd2 = pd;
    rot = Table[0, {Length[skel]}];
    place[i_, a_] := Module[
      {ni, na, arc, dir, oparc},
      arc = pd2[[i, a]];
      {{ni, na}} = Complement[Position[pd2, arc], {{i, a}}];
      If[emb[[ni]] === Null,
       emb[[ni]] = 3 - a + emb[[i]];
       pd2[[ni]] = RotateLeft[pd1[[ni]], na - 1];
       place[ni, #] & /@ {2, 3, 4},
       (* Else *) oparc = RotateLeft[pd2[[i]], 2][[a]];
       If[done[[arc]] === Null,
        done[[arc]] = 1;
        dir = If[arc - oparc == 1 || arc - oparc < -1, 1, -1];
        rot[[Position[skel, arc][[1, 1]]]] += dir * (emb[[ni]] - emb[[i]] + a - na - 2)
       ]
      ]
     ];
    place[1, #] & /@ {1, 2, 3, 4};
    k = -rot / 4;
    For[j = 1, j ≤ l, j++,
     k = ReplacePart[k,
       -1 + k[[Position[skel, pd[[j, 1]]][[1, 1]]]], Position[skel, pd[[j, 1]]][[1, 1]]]
    ];
    For[i = 1, i ≤ Length[k], i++,
     M *= t[i]^((1 / 2) * k[[i]])
    ];
    If[pd[[1, 4]] == pd[[1, 2]] + 1 || pd[[1, 2]] - pd[[1, 4]] > 1,
     M *= t[Position[skel, pd[[1, 1]]][[1, 1]]] * t[Position[skel, pd[[1, 2]]][[1, 1]]],
     M *= t[Position[skel, pd[[1, 1]]][[1, 1]]]
    ];
    Evaluate[M /. t → #] &
   ]
```

```
MV = MultivariableAlexander; MV2 = MultivariableAlexander2
```

MultivariableAlexander2

```
test1[L_] := (
    mv = MV[L][t]; mv2 = First[MV2[L][t]];
    Or @@ Map[
      (mv1 = mv /. t[i_] :→ t[#[[i]]]; Head[Expand[Simplify[mv2 / mv1]]] =!= Plus) &,
      Permutations[Range[Length[Skeleton[L]]]]
     ]
   );
Print[# -> test1[#]] & /@ AllLinks[9];
```

KnotTheory::loading : Loading precomputed data in MultivariableAlexander4Links`.

KnotTheory::loading : Loading precomputed data in PD4Links`.

```
Link[9, Alternating, 1] → True
Link[9, Alternating, 2] → True
Link[9, Alternating, 3] → True
Link[9, Alternating, 4] → True
Link[9, Alternating, 5] → True
Link[9, Alternating, 6] → True
Link[9, Alternating, 7] → True
Link[9, Alternating, 8] → True
Link[9, Alternating, 9] → True
Link[9, Alternating, 10] → True
Link[9, Alternating, 11] → True
Link[9, Alternating, 12] → True
Link[9, Alternating, 13] → True
Link[9, Alternating, 14] → True
Link[9, Alternating, 15] → True
Link[9, Alternating, 16] → True
Link[9, Alternating, 17] → True
Link[9, Alternating, 18] → True
Link[9, Alternating, 19] → True
Link[9, Alternating, 20] → True
Link[9, Alternating, 21] → True
Link[9, Alternating, 22] → True
Link[9, Alternating, 23] → True
Link[9, Alternating, 24] → True
Link[9, Alternating, 25] → True
Link[9, Alternating, 26] → True
Link[9, Alternating, 27] → True
Link[9, Alternating, 28] → True
Link[9, Alternating, 29] → True
Link[9, Alternating, 30] → True
Link[9, Alternating, 31] → True
Link[9, Alternating, 32] → True
Link[9, Alternating, 33] → True
Link[9, Alternating, 34] → True
```

Link[9, Alternating, 35] → True

Link[9, Alternating, 36] → True

Link[9, Alternating, 37] → True

Link[9, Alternating, 38] → True

Link[9, Alternating, 39] → True

Link[9, Alternating, 40] → True

Link[9, Alternating, 41] → True

Link[9, Alternating, 42] → True

Link[9, Alternating, 43] → True

Link[9, Alternating, 44] → True

Link[9, Alternating, 45] → True

Link[9, Alternating, 46] → True

Link[9, Alternating, 47] → True

Link[9, Alternating, 48] → True

Link[9, Alternating, 49] → True

Link[9, Alternating, 50] → True

Link[9, Alternating, 51] → True

Link[9, Alternating, 52] → True

Link[9, Alternating, 53] → True

Link[9, Alternating, 54] → True

Link[9, Alternating, 55] → True

Link[9, NonAlternating, 1] → True

Link[9, NonAlternating, 2] → True

Link[9, NonAlternating, 3] → True

Link[9, NonAlternating, 4] → True

Link[9, NonAlternating, 5] → True

Link[9, NonAlternating, 6] → True

Link[9, NonAlternating, 7] → True

Link[9, NonAlternating, 8] → True

Link[9, NonAlternating, 9] → True

Link[9, NonAlternating, 10] → True

Link[9, NonAlternating, 11] → True

Link[9, NonAlternating, 12] → True

Link[9, NonAlternating, 13] → True

Link[9, NonAlternating, 14] → True

```
Link[9, NonAlternating, 15] → True
Link[9, NonAlternating, 16] → True
Link[9, NonAlternating, 17] → True
Link[9, NonAlternating, 18] → True
Link[9, NonAlternating, 19] → True
Link[9, NonAlternating, 20] → True
Link[9, NonAlternating, 21] → True
Link[9, NonAlternating, 22] → True
Link[9, NonAlternating, 23] → True
Link[9, NonAlternating, 24] → True
Link[9, NonAlternating, 25] → True
Link[9, NonAlternating, 26] → True
```

First::normal : Nonatomic expression expected at position 1 in First[0]. ≫

Power::infy : Infinite expression $\frac{1}{0}$ encountered. ≫

Power::infy : Infinite expression $\frac{1}{0}$ encountered. ≫

Power::infy : Infinite expression $\frac{1}{0}$ encountered. ≫

General::stop : Further output of Power::infy will be suppressed during this calculation. ≫

```
Link[9, NonAlternating, 27] → True
Link[9, NonAlternating, 28] → True
```

```
{MV[Link[9, NonAlternating, 27]][t], MV2[Link[9, NonAlternating, 27]][t]}
```

```
{0, 0}
```

```
Flip[X[i_, j_, k_, l_]] := If[l == j + 1 || j - l > 1, X[j, k, l, i], X[l, i, j, k]];
VCube[pd_, l_List] := Module[
  {f},
  Expand[pd * Times @@ ((1 - f[#]) & /@ l)] //. pd1_PD * f[i_] :> MapAt[Flip, pd1, i]
 ]
```

```
Series[VCube[PD[#], {1, 2, 7}] /. pd_PD :> Jones[pd][E^x], {x, 0, 3}] & /@ AllLinks[8]
```

$\{-9 x^3 + O[x]^4, -12 x^3 + O[x]^4, 12 x^3 + O[x]^4, -12 x^3 + O[x]^4, 12 x^3 + O[x]^4, 12 x^3 + O[x]^4,$
$3 x^3 + O[x]^4, -15 x^3 + O[x]^4, O[x]^4, 12 x^3 + O[x]^4, 15 x^3 + O[x]^4, 12 x^3 + O[x]^4,$
$12 x^3 + O[x]^4, 15 x^3 + O[x]^4, -18 x^3 + O[x]^4, -18 x^3 + O[x]^4, -18 x^3 + O[x]^4, 18 x^3 + O[x]^4,$
$18 x^3 + O[x]^4, -24 x^3 + O[x]^4, 36 x^3 + O[x]^4, 9 x^3 + O[x]^4, 9 x^3 + O[x]^4, -18 x^3 + O[x]^4,$
$-18 x^3 + O[x]^4, -18 x^3 + O[x]^4, -24 x^3 + O[x]^4, -36 x^3 + O[x]^4, -36 x^3 + O[x]^4\}$

```
Print[# → Series[VCube[PD[#], {1, 2, 7}] /. pd_PD :> MV2[pd][t] /. t[i_] → E^(h x[i]),
      {h, 0, 3}]] & /@ AllLinks[8];
```

Link[8, Alternating, 1] $\to -x[2]^2 h^2 + \left(x[1]^2 x[2] - x[2]^3\right) h^3 + O[h]^4$

Link[8, Alternating, 2] $\to$

$-x[2]\,h + \frac{1}{4}\left(-3\,x[1]^2 + 2\,x[1]\,x[2] + 3\,x[2]^2\right) h^2 + \left(\frac{1}{4}\,x[1]^2\,x[2] + \frac{7\,x[2]^3}{12}\right) h^3 + O[h]^4$

Link[8, Alternating, 3] $\to \frac{3}{4}\left(x[1]^2 - 2\,x[1]\,x[2] + x[2]^2\right) h^2 + O[h]^4$

Link[8, Alternating, 4] $\to$

$\frac{1}{4}\left(-3\,x[1]^2 - 2\,x[1]\,x[2] + 5\,x[2]^2\right) h^2 + \frac{1}{4}\left(3\,x[1]^2\,x[2] - 2\,x[1]\,x[2]^2 - x[2]^3\right) h^3 + O[h]^4$

Link[8, Alternating, 5] $\to$

$x[2]\,h + \frac{1}{4}\left(3\,x[1]^2 - 2\,x[1]\,x[2] - 3\,x[2]^2\right) h^2 + \left(-\frac{1}{4}\,x[1]^2\,x[2] - 2\,x[1]\,x[2]^2 + \frac{17\,x[2]^3}{12}\right) h^3 + O[h]^4$

Link[8, Alternating, 6] $\to$

$x[2]\,h + \frac{1}{4}\left(3\,x[1]^2 - 2\,x[1]\,x[2] - 3\,x[2]^2\right) h^2 + \left(-\frac{1}{4}\,x[1]^2\,x[2] - 2\,x[1]\,x[2]^2 + \frac{5\,x[2]^3}{12}\right) h^3 + O[h]^4$

Link[8, Alternating, 7] $\to -x[1]\,x[2]\,h^2 + \left(-x[1]^2\,x[2] + x[2]^3\right) h^3 + O[h]^4$

Link[8, Alternating, 8] $\to$

$\frac{1}{4}\left(-3\,x[1]^2 + 10\,x[1]\,x[2] - 3\,x[2]^2\right) h^2 + \frac{1}{4}\left(3\,x[1]^3 - 2\,x[1]^2\,x[2] - 5\,x[1]\,x[2]^2\right) h^3 + O[h]^4$

Link[8, Alternating, 9] $\to -x[1]\,x[2]^2\,h^3 + O[h]^4$

Link[8, Alternating, 10] $\to$

$\frac{3}{4}\left(x[1]^2 - 2\,x[1]\,x[2] + x[2]^2\right) h^2 + \frac{3}{4}\left(x[1]^3 - 3\,x[1]^2\,x[2] + 3\,x[1]\,x[2]^2 - x[2]^3\right) h^3 + O[h]^4$

Link[8, Alternating, 11] $\to$

$\frac{1}{4}\left(3\,x[1]^2 - 10\,x[1]\,x[2] + 3\,x[2]^2\right) h^2 + \frac{1}{4}\left(-3\,x[1]^3 + 13\,x[1]^2\,x[2] - 13\,x[1]\,x[2]^2 + 3\,x[2]^3\right) h^3 + O[h]^4$

Link[8, Alternating, 12] $\to$

$\frac{3}{4}\left(x[1]^2 - 2\,x[1]\,x[2] + x[2]^2\right) h^2 + \frac{3}{2}\left(x[1]^3 - 3\,x[1]^2\,x[2] + 3\,x[1]\,x[2]^2 - x[2]^3\right) h^3 + O[h]^4$

Link[8, Alternating, 13] $\to$

$\frac{3}{4}\left(x[1]^2 - 2\,x[1]\,x[2] + x[2]^2\right) h^2 + \frac{3}{2}\left(x[1]^3 - 3\,x[1]^2\,x[2] + 3\,x[1]\,x[2]^2 - x[2]^3\right) h^3 + O[h]^4$

Link[8, Alternating, 14] $\to$

$\frac{1}{4}\left(3\,x[1]^2 - 10\,x[1]\,x[2] + 3\,x[2]^2\right) h^2 + \frac{1}{2}\left(-3\,x[1]^3 + 13\,x[1]^2\,x[2] - 13\,x[1]\,x[2]^2 + 3\,x[2]^3\right) h^3 + O[h]^4$

Link[8, Alternating, 15] $\to -\frac{3}{4}\left(x[1]^2\,x[3] - 2\,x[1]\,x[2]\,x[3] + x[2]^2\,x[3]\right) h^3 + O[h]^4$

Link[8, Alternating, 16] $\to O[h]^4$

Link[8, Alternating, 17] $\to -\frac{3}{2}\left(x[1]^2\,x[3] - 2\,x[1]\,x[2]\,x[3] + x[2]^2\,x[3]\right) h^3 + O[h]^4$

$\text{Link}[8, \text{Alternating}, 18] \to -2\, x[3]^2\, h^2 + \left(-\frac{3}{2}\, x[1]^2\, x[3] + 3\, x[1]\, x[2]\, x[3] - \frac{3}{2}\, x[2]^2\, x[3] - 2\, x[1]\, x[3]^2 + 2\, x[2]\, x[3]^2 + x[3]^3\right) h^3 + O[h]^4$

$\text{Link}[8, \text{Alternating}, 19] \to -x[1]^2\, h^2 + \left(-\frac{1}{2}\, x[1]^3 + x[1]^2\, x[3] - x[1]\, x[2]\, x[3]\right) h^3 + O[h]^4$

$\text{Link}[8, \text{Alternating}, 20] \to 2\, x[2]\, x[3]\, h^2 + \left(\frac{3}{2}\, x[1]^2\, x[3] + 2\, x[1]\, x[2]\, x[3] - x[2]^2\, x[3] - 3\, x[1]\, x[3]^2 - 2\, x[2]\, x[3]^2 + \frac{3\, x[3]^3}{2}\right) h^3 + O[h]^4$

ReplacePart::partw : Part {5, 9} of
{{1, 0, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 1, 0, 0, 0}, {0, 0, 0, 0, 1, 0, 0, 0}, ≪3≫, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, ≪6≫} does not exist. ≫

ReplacePart::partw : Part {6, 9} of
ReplacePart[{{1, 0, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 1, 0, 0, 0}, {0, 0, 0, 0, 1, 0, 0, 0}, ≪4≫, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, ≪6≫}, 1, {5, 9}] does not exist. ≫

ReplacePart::partw : Part {7, 9} of
ReplacePart[ReplacePart[{{1, 0, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 1, 0, 0, 0}, ≪5≫, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, ≪6≫}, 1, {5, 9}], 1, {6, 9}] does not exist. ≫

General::stop : Further output of ReplacePart::partw will be suppressed during this calculation. ≫

Transpose::argt : Transpose called with 0 arguments; 1 or 2 arguments are expected. ≫

$$\text{Link}[8, \text{Alternating}, 21] \to \frac{\text{Det}[\text{Transpose}[]]}{x[1]\, h} - \frac{\text{Det}[\text{Transpose}[]]\,(3\, x[1] - x[2] + x[3] + 2\, x[4])}{2\, x[1]} + \frac{1}{24\, x[1]} \text{Det}[\text{Transpose}[]] \left(26\, x[1]^2 - 18\, x[1]\, x[2] + 3\, x[2]^2 + 18\, x[1]\, x[3] - 6\, x[2]\, x[3] + 3\, x[3]^2 + 36\, x[1]\, x[4] - 12\, x[2]\, x[4] + 12\, x[3]\, x[4] + 12\, x[4]^2\right) h + \frac{1}{48\, x[1]} \text{Det}[\text{Transpose}[]] \left(-24\, x[1]^3 + 26\, x[1]^2\, x[2] - 9\, x[1]\, x[2]^2 + x[2]^3 - 26\, x[1]^2\, x[3] + 18\, x[1]\, x[2]\, x[3] - 3\, x[2]^2\, x[3] - 9\, x[1]\, x[3]^2 + 3\, x[2]\, x[3]^2 - x[3]^3 - 52\, x[1]^2\, x[4] + 36\, x[1]\, x[2]\, x[4] - 6\, x[2]^2\, x[4] - 36\, x[1]\, x[3]\, x[4] + 12\, x[2]\, x[3]\, x[4] - 6\, x[3]^2\, x[4] - 36\, x[1]\, x[4]^2 + 12\, x[2]\, x[4]^2 - 12\, x[3]\, x[4]^2 - 8\, x[4]^3\right) h^2 + \frac{1}{5760\, x[1]} \text{Det}[\text{Transpose}[]] \left(952\, x[1]^4 - 1440\, x[1]^3\, x[2] + 780\, x[1]^2\, x[2]^2 - 180\, x[1]\, x[2]^3 + 15\, x[2]^4 + 1440\, x[1]^3\, x[3] - 1560\, x[1]^2\, x[2]\, x[3] + 540\, x[1]\, x[2]^2\, x[3] - 60\, x[2]^3\, x[3] + 780\, x[1]^2\, x[3]^2 - 540\, x[1]\, x[2]\, x[3]^2 + 90\, x[2]^2\, x[3]^2 + 180\, x[1]\, x[3]^3 - 60\, x[2]\, x[3]^3 + 15\, x[3]^4 + 2880\, x[1]^3\, x[4] - 3120\, x[1]^2\, x[2]\, x[4] + 1080\, x[1]\, x[2]^2\, x[4] - 120\, x[2]^3\, x[4] + 3120\, x[1]^2\, x[3]\, x[4] - 2160\, x[1]\, x[2]\, x[3]\, x[4] + 360\, x[2]^2\, x[3]\, x[4] + 1080\, x[1]\, x[3]^2\, x[4] - 360\, x[2]\, x[3]^2\, x[4] + 120\, x[3]^3\, x[4] + 3120\, x[1]^2\, x[4]^2 - 2160\, x[1]\, x[2]\, x[4]^2 + 360\, x[2]^2\, x[4]^2 + 2160\, x[1]\, x[3]\, x[4]^2 - 720\, x[2]\, x[3]\, x[4]^2 + 360\, x[3]^2\, x[4]^2 + 1440\, x[1]\, x[4]^3 - 480\, x[2]\, x[4]^3 + 480\, x[3]\, x[4]^3 + 240\, x[4]^4\right) h^3 + O[h]^4$$

$$\text{Link}[8, \text{NonAlternating}, 1] \to x[2]^2\, h^2 + \left(3\, x[1]\, x[2]^2 - 3\, x[2]^3\right) h^3 + O[h]^4$$

$$\text{Link}[8, \text{NonAlternating}, 2] \to x[2]^2\, h^2 + \left(-x[1]\, x[2]^2 + 2\, x[2]^3\right) h^3 + O[h]^4$$

$$\text{Link}[8, \text{NonAlternating}, 3] \to \frac{1}{4} \left(-3\, x[1]\, x[2]^2 - 3\, x[2]^3 + 10\, x[1]\, x[2]\, x[3] + 10\, x[2]^2\, x[3] - 3\, x[1]\, x[3]^2 - 3\, x[2]\, x[3]^2\right) h^3 + O[h]^4$$

$$\text{Link}[8, \text{NonAlternating}, 4] \to \frac{1}{4} \left(3\, x[1]\, x[2]^2 - 3\, x[2]^3 - 10\, x[1]\, x[2]\, x[3] + 10\, x[2]^2\, x[3] + 3\, x[1]\, x[3]^2 - 3\, x[2]\, x[3]^2\right) h^3 + O[h]^4$$

$$\text{Link}[8, \text{NonAlternating}, 5] \to \frac{1}{4} \left(-3\, x[2]^3 + 10\, x[2]^2\, x[3] - 3\, x[2]\, x[3]^2\right) h^3 + O[h]^4$$

$$\text{Link}[8, \text{NonAlternating}, 6] \to -2\,(x[2]\, x[3])\, h^2 + \left(-\frac{3}{2} x[1]^2\, x[3] - 2\, x[1]\, x[2]\, x[3] + x[2]^2\, x[3] + 3\, x[1]\, x[3]^2 + 2\, x[2]\, x[3]^2 - \frac{3\, x[3]^3}{2}\right) h^3 + O[h]^4$$

$$\text{Link}[8, \text{NonAlternating}, 7] \to O[h]^4$$

$$\text{Link}[8, \text{NonAlternating}, 8] \to O[h]^4$$