```
<< KnotTheory`
```

```
Loading KnotTheory` version of July 7, 2007, 9:53:56.5473.
Read more at http://katlas.math.toronto.edu/wiki/KnotTheory.
```

```
BeginPackage["KnotTheory`"];

MultivariableAlexander2[PD[Loop[_]]] := {1} &
MultivariableAlexander2[K_] /; Head[K] =!= PD := MultivariableAlexander2[PD[K]]


MultivariableAlexander2[pd_PD] := (MultivariableAlexander2[pd] =
    (l = Length[pd]; mat = Table[0, {2 * l}, {2 * l}]; skel = Skeleton[pd]; pd1 = List @@ pd;
     G = Table[0, {2 * l}, {l}]; pd1 //. X[a_, b_, c_, d_] :> If[d == b + 1 || b - d > 1,
        {mat[[c, a]] = -t[b]; mat[[c, b]] = t[a] - 1; mat[[c, c]] = 1},
        {mat[[c, a]] = -1; mat[[c, b]] = 1 - t[a] ; mat[[c, c]] = t[b]}];
     c = Times @@ pd /. {X[i_, j_, k_, l_] /; (l - j == 1 || j - l > 1) :> path[k] path[i] path[j, l],
         X[i_, j_, k_, l_] /; (j - l == 1 || l - j > 1) :> path[k] path[i] path[l, j],
         P[i_, j_] :> path[i, j]} //. {path[a__, i_] path[i_, b__] :> path[a, i, b],
        path[a__, i_] path[b__, i_] :> Join[path[a, i], Reverse[path[b]]],
        path[i_, a__] path[i_, b__] :> Join[Reverse[path[b]], path[i, a]],
        path[a__, i_] path[i_] :> path[a, i],
        path[i_, a__] path[i_] :> path[a, i], path[i_] path[i_] :> path[i]};
     For[i = 1, i <= 2 * l, i++, G = ReplacePart[G, 1, {i, First[First[Position[c, i]]]}]];
     mat = mat /. t[a_] :> t[Position[skel, a][[1, 1]]];
     M = Factor[Simplify[Det[Delete[Transpose[Delete[Transpose[G].mat.G,
              Position[c, pd1[[1, 1]]][[1, 1]]]], Position[c, pd1[[1, 1]]][[1, 1]]]] /
         (t[Position[skel, pd1[[1, 1]]][[1, 1]]] - 1)]]; emb = Table[Null, {Length[pd]}];
     done = Table[Null, {2 * Length[pd]}]; emb[[1]] = 0; pd2 = pd;
     rot = Table[0, {Length[skel]}]; place[i_, a_] := Module[
       {ni, na, arc, dir, oparc},
       arc = pd2[[i, a]];
       {{ni, na}} = Complement[Position[pd2, arc], {{i, a}}];
       If[emb[[ni]] === Null,
        emb[[ni]] = 3 - a + emb[[i]];
        pd2[[ni]] = RotateLeft[pd1[[ni]], na - 1];
        place[ni, #] & /@ {2, 3, 4},
        (* Else *) oparc = RotateLeft[pd2[[i]], 2][[a]];
        If[done[[arc]] === Null,
         done[[arc]] = 1;
         dir = If[arc - oparc == 1 || arc - oparc < -1, 1, -1];
         rot[[Position[skel, arc][[1, 1]]]] += dir * (emb[[ni]] - emb[[i]] + a - na - 2)]]];
     place[1, #] & /@ {1, 2, 3, 4}; k = -rot / 4; For[j = 1, j <= l, j++, k = ReplacePart[k,
        -1 + k[[Position[skel, pd[[j, 1]]][[1, 1]]]], Position[skel, pd[[j, 1]]][[1, 1]]]];
     For[i = 1, i <= Length[k], i++, M = t[i]^{(1 / 2) * k[[i]]} * M];
     If[pd[[1, 4]] == pd[[1, 2]] + 1 || pd[[1, 2]] - pd[[1, 4]] > 1,
      M = M * t[Position[skel, pd[[1, 1]]][[1, 1]]] * t[Position[skel, pd[[1, 2]]][[1, 1]]],
      M = M * t[Position[skel, pd[[1, 1]]][[1, 1]]]];
     Evaluate[M /. t -> #] &))

MV = MultivariableAlexander; MV2 = MultivariableAlexander2
```

```
MultivariableAlexander2
```

```
MV2[Link["L5a1"]][t]
```

$$\left\{\frac{(-1+t[1])\,(-1+t[2])}{\sqrt{t[1]}\,\sqrt{t[2]}}\right\}$$

```
MV[Link["L5a1"]][t]
```

KnotTheory::loading : Loading precomputed data in MultivariableAlexander4Links`.

$$-1+t[1]+t[2]-t[1]\,t[2]$$

```
test[L_] := Head[Expand[Simplify[MV2[L][t][[1]] / MV[L][t]]]] =!= Plus
```

```
test1[L_] := (
  mv = MV[L][t]; mv2 = First[MV2[L][t]];
  Or @@ Map[
    (mv1 = mv /. t[i_] :> t[#[[i]]]; Head[Expand[Simplify[mv2 / mv1]]] =!= Plus) &,
    Permutations[Range[Length[Skeleton[L]]]]
   ]
 )
```

```
test1[Link["L5a1"]]
```

True

```
test1 /@ AllLinks[9]
```

Power::infy : Infinite expression $\frac{1}{0}$ encountered. ≫

∞::indet : Indeterminate expression 0 ComplexInfinity encountered. ≫

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Power::infy : Infinite expression $\frac{1}{0}$ encountered. ≫

General::stop : Further output of Power::infy will be suppressed during this calculation. ≫

∞::indet : Indeterminate expression 0 ComplexInfinity encountered. ≫

General::stop : Further output of ∞::indet will be suppressed during this calculation. ≫

```
{True, True, True, True, True, True, True, True, True, True, True, True,
 True, True, True, True, True, True, True, True, True, True, True, True, True,
 True, True, True, True, True, True, True, True, True, True, True, True, True,
 True, True, True, True, True, True, True, True, True, True, True, True, True,
 True, True, True, True, True, True, True, True, True, True, True, True, True,
 True, True, True, True, True, True, True, True, True, True, True, True, True}
```

```
Timing[MV[PD[#]][t] & /@ AllLinks[10]]
```

KnotTheory::credits :

Vogel's algorithm was implemented by Dan Carney in the summer of 2005 at the University of Toronto.

KnotTheory::credits :

The multivariable Alexander program was written by Dan Carney at the University of Toronto in the summer of 2005.

A very large output was generated. Here is a sample of it:

```
{118.141,
 {-1 + 4 t[1] - 8 t[1]^2 + 8 t[1]^3 - 4 t[1]^4 + t[1]^5 + t[2] - 4 t[1] t[2] + 8 t[1]^2 t[2] - 8 t[1]^3 t[2] +
   4 t[1]^4 t[2] - t[1]^5 t[2], -1 + 4 t[1] - 6 t[1]^2 + 6 t[1]^3 - 4 t[1]^4 + t[1]^5 + t[2] - 4 t[1] t[2] +
   6 t[1]^2 t[2] - 6 t[1]^3 t[2] + 4 t[1]^4 t[2] - t[1]^5 t[2], <<284>>,
  t[2] t[3] - t[1] t[2] t[3] + t[1] t[4] - t[1] t[2] t[4] - t[3] t[4] + t[1] t[2] t[3] t[4] -
   t[5] + t[1] t[2] t[5] + t[3] t[5] - t[2] t[3] t[5] + t[4] t[5] - t[1] t[4] t[5]}}
```

Show Less | Show More | Show Full Output | Set Size Limit...

```
Timing[MV2[PD[#]][t] & /@ AllLinks[10]]
```

A very large output was generated. Here is a sample of it:

$$\left\{16.25,\ \left\{\left\{\frac{(-1+t[1])\sqrt{t[1]}\ (-1+t[2])\ \left(1-3t[2]+5t[2]^2-3t[2]^3+t[2]^4\right)}{t[2]^{3/2}}\right\},\right.\right.$$
$$\left\{-\frac{(-1+t[1])\sqrt{t[1]}\ (-1+t[2])\ \left(1-3t[2]+3t[2]^2-3t[2]^3+t[2]^4\right)}{t[2]^{7/2}}\right\},\ \ll 284 \gg,$$
$$\left.\left\{\left(t[1]\,t[2]-t[3]+t[2]\,t[3]-t[1]\,t[2]\,t[3]-t[2]\,t[4]+t[3]\,t[4]-t[1]\,t[2]\,t[5]+\right.\right.\right.$$
$$\left.\left.\left.t[1]\,t[3]\,t[5]+t[4]\,t[5]-t[1]\,t[4]\,t[5]+t[1]\,t[2]\,t[4]\,t[5]-t[3]\,t[4]\,t[5]\right)\Big/\left(\sqrt{t[1]}\ \sqrt{t[2]}\ \sqrt{t[3]}\ \sqrt{t[4]}\ \sqrt{t[5]}\right)\right\}\right\}\right\}$$

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```
MV3[L_] :=
 Expand[PowerExpand[MV2[L][t][[1]] * Sqrt[Product[t[i], {i, 1, Length[Skeleton[L]]}]]]]
```

```
MV2[AllLinks[7][[5]]][t]
```

$$\left\{\frac{(-1+t[1]+t[2])\ (-t[1]-t[2]+t[1]\,t[2])}{t[1]\,t[2]}\right\}$$

```
Sqrt[t[1]] / Sqrt[t[1]]
```

```
1
```

```
Flip[X[i_, j_, k_, l_]] := If[l == j + 1 || j - l > 1, X[j, k, l, i], X[l, i, j, k]]
```

```
VCube[pd_, l_List] := Module[
  {f},
  Expand[pd * Times @@ ((1 - f[#]) & /@ l)] //. pd1_PD * f[i_] :> MapAt[Flip, pd1, i]
 ]
```

```
pd = PD[Knot[3, 1]]
```

KnotTheory::loading : Loading precomputed data in PD4Knots`.

```
PD[X[1, 4, 2, 5], X[3, 6, 4, 1], X[5, 2, 6, 3]]
```

```
Series[VCube[PD[Knot[8, 17]], {1, 3, 6}] /. pd_PD :> Jones[pd][E^x], {x, 0, 5}]
```

$$-12\,x^3+15\,x^4+11\,x^5+O[x]^6$$

```
Series[VCube[PD[#], {1, 2, 7}] /. pd_PD :> MV2[pd][t] /. t[i_] → E^(h x[i]), {h, 0, 3}] & /@
 AllLinks[8]
```

ReplacePart::partw : Part {5, 9} of

{{1, 0, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 1, 0, 0, 0}, ≪5≫, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, ≪6≫} does not exist. ≫

ReplacePart::partw : Part {6, 9} of

ReplacePart[{{1, 0, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 1, 0, 0, 0}, ≪6≫, {0, 0, 0, 0, 0, 0, 0, 0}, ≪6≫}, 1, {5, 9}] does not exist. ≫

ReplacePart::partw : Part {7, 9} of

ReplacePart[ReplacePart[{{1, 0, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0, 0}, ≪7≫, {0, 0, 0, 0, 0, 0, 0, 0}, ≪6≫}, 1, {5, 9}], 1, {6, 9}] does not exist. ≫

General::stop : Further output of ReplacePart::partw will be suppressed during this calculation. ≫

Transpose::argt : Transpose called with 0 arguments; 1 or 2 arguments are expected. ≫

$$\Big\{\{-x[3]^2\,h^2+O[h]^4\},\ \Big\{-x[3]\,h+\frac{1}{2}x[3]^2\,h^2+\frac{5}{6}x[3]^3\,h^3+O[h]^4\Big\},\ \{O[h]^4\},\ \{O[h]^4\},$$
$$\Big\{x[3]\,h-\frac{1}{2}x[3]^2\,h^2-\frac{5}{6}x[3]^3\,h^3+O[h]^4\Big\},\ \Big\{x[3]\,h-\frac{1}{2}x[3]^2\,h^2-\frac{11}{6}x[3]^3\,h^3+O[h]^4\Big\},$$
$$\{-x[3]^2\,h^2+O[h]^4\},\ \{x[3]^2\,h^2-x[3]^3\,h^3+O[h]^4\},\ \{-x[3]^3\,h^3+O[h]^4\},\ \{O[h]^4\},$$
$$\{-x[3]^2\,h^2+O[h]^4\},\ \{O[h]^4\},\ \{O[h]^4\},\ \{-x[3]^2\,h^2+O[h]^4\},\ \{O[h]^4\},\ \{O[h]^4\},\ \{O[h]^4\},$$
$$\Big\{-2\,x[4]^2\,h^2+x[4]^3\,h^3+O[h]^4\Big\},\ \Big\{-x[4]^2\,h^2-\frac{1}{2}x[4]^3\,h^3+O[h]^4\Big\},\ \Big\{2\,x[4]^2\,h^2-x[4]^3\,h^3+O[h]^4\Big\},$$
$$\Big\{\frac{\text{Det}[\text{Transpose}[]]}{x[5]\,h}-\frac{5}{2}\,\text{Det}[\text{Transpose}[]]+\frac{37}{12}\,\text{Det}[\text{Transpose}[]]\,x[5]\,h-$$
$$\frac{5}{2}\left(\text{Det}[\text{Transpose}[]]\,x[5]^2\right)h^2+\frac{1079}{720}\,\text{Det}[\text{Transpose}[]]\,x[5]^3\,h^3+O[h]^4\Big\},$$
$$\{x[3]^2\,h^2+O[h]^4\},\ \{x[3]^2\,h^2+x[3]^3\,h^3+O[h]^4\},\ \{2\,x[4]^3\,h^3+O[h]^4\},\ \{O[h]^4\},$$
$$\{x[4]^3\,h^3+O[h]^4\},\ \{-2\,x[4]^2\,h^2+x[4]^3\,h^3+O[h]^4\},\ \{O[h]^4\},\ \{O[h]^4\}\Big\}$$

```
Series[VCube[PD[#], {1, 2, 7, 9}] /. pd_PD :> MV2[pd][t] /. t[i_] → E^(h x[i]),
   {h, 0, 4}] & /@ AllLinks[9]
```

ReplacePart::partw : Part {5, 10} of
{{1, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 1, 0, 0, 0, 0}, ≪5≫, {0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0}, ≪8≫} does not exist. ≫

ReplacePart::partw : Part {6, 10} of ReplacePart[≪1≫] does not exist. ≫

ReplacePart::partw : Part {7, 10} of ReplacePart[≪1≫] does not exist. ≫

General::stop : Further output of ReplacePart::partw will be suppressed during this calculation. ≫

Transpose::argt : Transpose called with 0 arguments; 1 or 2 arguments are expected. ≫

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Transpose::argt : Transpose called with 0 arguments; 1 or 2 arguments are expected. ≫

General::stop : Further output of Transpose::argt will be suppressed during this calculation. ≫

Delete::partw : Part {7} of
Transpose[ReplacePart[≪1≫, 1, {18, 6}]].{{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, ≪8≫}, ≪9≫, ≪8≫}.ReplacePart[ReplacePart[≪1≫], 1, {18, 6}] does not exist. ≫

Delete::partw : Part {7} of
Transpose[Delete[Transpose[≪1≫].{{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, ≪8≫}, ≪9≫, ≪8≫}.ReplacePart[ReplacePart[≪1≫], 1, {18, 6}], 7]] does not exist. ≫

Delete::partw : Part {7} of
Transpose[ReplacePart[≪1≫, 1, {18, 6}]].{{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, ≪8≫}, ≪9≫, ≪8≫}.ReplacePart[ReplacePart[≪1≫], 1, {18, 6}] does not exist. ≫

General::stop : Further output of Delete::partw will be suppressed during this calculation. ≫

```
$Aborted
```

```
Series[VCube[PD[Link["L11n300"]], {1, 2, 7}] /. pd_PD :> MV[pd][t] /. t[i_] → E^(h x[i]),
 {h, 0, 3}]
```

$$-4\,x[4]\,h-3\,x[4]^2\,h^2-\frac{5}{3}\,x[4]^3\,h^3+O[h]^4$$