PSU Talk

January-26-09
8:58 AM

(u, v, and w knots) x (topology, combinatorics, low algebra, and high algebra)

[Department of Mathematics Colloquium](http://www.math.toronto.edu/~drorbn/Talks/PSU-090205/)

Penn State University, February 5, 2009

Abstract. My subject is a Cartesian product. It runs in three parallel columns - the u column, for usual knots, the v column, for virtual knots, and the w column, for welded, or weakly virtual, or warmup knots. Each class of knots has a topological meaning and a "finite type" theory, which leads to some combinatorics, somewhat different combinatorics in each case. In each column the resulting combinatorics ends up describing tensors within a different "low algebra" universe - the universe of metrized Lie algebras for u, the richer universe of Lie bialgebras for v, and for w, the wider and therefore less refined universe of general Lie algebras. In each column there is a "fundamental theorem" to be proven (or conjectured), and the means, in each column, is a different piece of "high algebra": associators and quasi-Hopf algebras in one, deformation quantization à la Etingof and Kazhdan in the second, and in the third, the Kashiwara-Vergne theory of convolutions on Lie groups. Finally, u maps to v and v maps to w at topology level, and the relationship persists and deepens the further down the columns one goes.

The 12 boxes in this product each deserves its own talk, and the few that are not yet fully understood deserve a few further years of research. Thus my talk will only give the flavour of a few of the boxes that I understand, and only hint at my expectations for the contents the (2,4) box, the one I understand the least and the one I wish to understand the most.

Pasted from <http://www.math.toronto.edu/~drorbn/Talks/PSU-090205/>

* use old tech for handout.

Title etc. 1. 5 mins

	u	→ v	→ w
topology	2. 5 mins	3. 5 mins	4. 5 mins
combinatorics	5. 10 mins	5.1 1 min	5.2 1 min
low algebra	7. 5 min	7.1 1 min	6. 4 min
high algebra	8. knots are the wrong object to study... 8 min	10. 2 min	9. 8 min

bold lines, not straight.

10"

7.5"

middle column least regular

Prep estimates:

Abstract/title	1hr
outline	1hr
~ done line	
Tech	1hr
Handout	4 hrs
printing etc.	2hrs.

(u, v, and w knots) x (topology, combinatorics, low algebra, and high algebra)

Dror Bar–Natan, Penn State February 5 2009, http://www.math.toronto.edu/~drorbn/Talks/PSU–090205/

"God created the knots, all else in topology is the work of mortals." Leopold Kronecker (modified)

1 u knots $\xrightarrow{1-1}$ v knots $\xrightarrow{\text{onto}}$ w knots

topology

2 Picture of first 48 knots

3 / 4 copy from Northeastern talk, first 2 rows.

combinatorics

5 given V:{knots}→A, extend using V(X) = V(X) − V(X)
"V on n-singular" = "an nth derivative of V"
Def V of type m ⟺ $V^{(m+1)} \equiv 0$ ("polynomial of degree m")
$V^{(m)}$ more or less determines V; $V^{(m)} \in A_m^*$ with
$A = \{\text{n chords}\}/4T$ 4T
need a "universal" Z:{knots}→A

6 V(X) = . . . V(X) = . . .
$A^v = \{\ \}/6T$
need a universal V:{virtual knots}→A^v

7 Same but TC & 4T replacing 6T, getting A^w

low algebra

10 similar, with metrized Lie algebras replacing arbitrary Lie algebras.

9 similar, with bialgs replacing Lie algs

8 Thm $A^w \cong A^{wt}$, where $A^{wt} := CA\langle Y, Y\rangle/\ldots = \{\ \}/\text{rels}$
"See Lie algebras":
Thm Given g, ∃ $T: A^w \to A^{wt} \to U(Ig)$

high algebra

11 knots are the wrong object to study in u-knot theory!
Not finitely generated; no interesting ops. 1350 algo.
KTGs | ops
Gens | rels
Thm: ∃ a homomorphic Z is the same as a "Drinfel'd associator"

13 "Z is a Quantum Group"; more precisely ought to be related to the Etingof–Kazhdan theory of quantization of Lie bialgebras.
Dror's Dream straighten & fatten this column

12 $wKTT = CA\langle\ , Y\rangle/K123,\ldots$
Thm(~) A homomorphic Z is equivalent to Kashiwara–Vergne
KV (proven by Alekseev–Meinrenken, 200..): For any f.d. Lie alg g,
$(\mathrm{Fun}(G)^{\mathrm{Ad}G}, *) \cong (\mathrm{Fun}(g)^{\mathrm{Ad}G}, \cdot)$
(closely related to "the orbit method")

McCool Goldsmith

[W] with chern-simons

Add a .png link.

Cattaneo BF

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"God created the knots, all else in topology is the work of mortals." Leopold Kronecker (modified)

1 u–knots $\xrightarrow{1-1}$ v–knots $\xrightarrow{\text{onto}}$ w–knots

topology

DBN, data by Brian Gilbert, http://www.math.toronto.edu/~drorbn/Gallery/KnottedObjects/Candies/

Pict of Reidemeister

u–knots are usual knots: 2

$\{\ \}/$ R1, R2, R3

$= PA\langle\ \mid R123\rangle$ o lugs

"Knots in $\mathbb{R}^3$"

v–knots are virtual knots: 3

$\{\ \}/$ R123, VR1, M

$= PA\langle\ \mid R123, VR123, M\rangle$ o

$= CA\langle\ \mid R123\rangle$ o

= Knots on surfaces, modulo stabilazation:

w is for welded, weakly v, and warmup:

4 {w–knots}={v–knots}/(OC)

where OC is Overcrossings Commute:

OC $\overset{w}{=}$ yet ≠ UC

Fenn Rimanyi Rourke Satoh Brendle Hatcher.

color in-t strands red & blue, orient black strand.

Related to "movies of flying rings" to knotted tubes in 4–space, and to "basis conjugating automorphisms of free groups".

picture of Kauffman

Picture of Polyak

combinatorics

Extend any V : {u-knots} → $\mathcal{A}$ to "singular u-knots" using $V(\times) := V(\times) - V(\times)$, and think "differentiation".
Declare "V is of type m" iff $V^{(m+1)} \equiv 0$, think "polynomial of degree m".
$W = V^{(m)}$ roughly determines V; $W \in \mathcal{A}_m^\star$ with

$\mathcal{A}_m := \{\ m \text{ chords}\ \}/ \overset{4T}{=}$

Need a "universal" Z : {u-knots} → $\mathcal{A} = \bigoplus \mathcal{A}_m$.

Vassiliev Goussarov

The Miller Institute knot 5

All the same, except 6

$V(\times) := V(\times) - V(\times)$

$V(\times) := V(\times) - V(\times)$

$\mathcal{A}^v :=$ {"arrow diagrams"}/6T

Need a Z : {v-knots} → $\mathcal{A}^v$.

All the same, except 7

$\mathcal{A}^w := \mathcal{A}^v/TC$

Need a Z : {w-knots} → $\mathcal{A}^w$.

"Tails Commute (TC)":

Flip order and move equal sign to the left, to accomodate 4T in the same picture.

low algebra

10 Similar with metrized Lie algebras replacing arbitrary Lie algebras

Pictures of Andrews, Drinfel'd Vogel

9 Similar with Lie bi–algebras replacing arbitrary Lie algebras

Theorem. $\mathcal{A}^w \cong \mathcal{A}^{wt} :=$ 8

$\{\ \text{only}\ \}/$ &TC

This screams, if you speak the language, Lie Algebras!
And indeed we have
Theorem. Given a finite dimensional Lie algebra $\mathfrak{g}$, there is $T : \mathcal{A}^w \to \mathcal{U}(I\mathfrak{g}) := \mathcal{U}(\mathfrak{g} \oplus \mathfrak{g}^*_{ab})$.

All capitals no exclamation

X

Knots are the wrong objects to study in knot theory! They are not finitely generated and they carry no interesting operations. 11

13 Z is a Quantum Group?

Switch to w-knotted trivalent tangles, 12
wKTT := $CA\langle\times,\times,Y\rangle$.

Picture

[W] with Chern-Simons

Add a .png link.

Cattaneo BF

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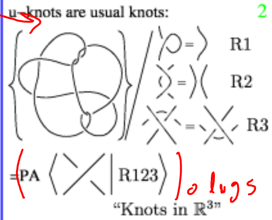
1 u–knots $\xrightarrow{1\text{–}1}$ v–knots $\xrightarrow{\text{onto}}$ w–knots

Pict of Reidemeister

topology

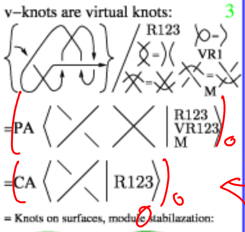
u–knots are usual knots: 2



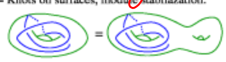
logs

"Knots in $\mathbb{R}^3$"

v–knots are virtual knots: 3



= Knots on surfaces, modulo stabilazation:



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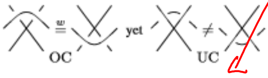
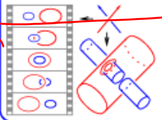


color input strands red & blue, orient black strand.



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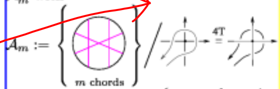

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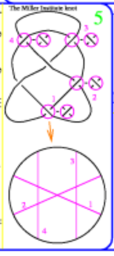
Vassiliev Goussarov



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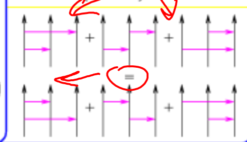

All the same, except

$\mathcal{A}^w := \mathcal{A}^v/TC$ 7

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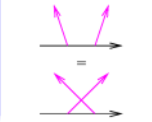
"Tails Commute (TC)":



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low algebra

10

Similar

with metrized Lie algebras replacing arbitrary Lie algebras

pictures of Anrose, Cvitanović, Vogel

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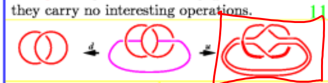
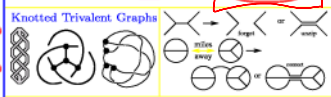
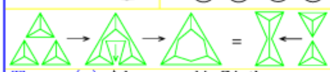
uniform line width.



Knotted Trivalent Graphs





Theorem (∼). A homomorphic Z is the same as a "Drinfel'd Associator".

13

Z is a Quantum Group?

More precisely, a homomorphic Z ought to be equivalent to the Etingof-Kazhdan theory of deformation quantization of Lie bialgebras.

Dror's Dream: Straighten and fatten this column.

Switch to w-knotted trivalent tangles, 12

$\text{wKTT} := CA\langle \times, \times, Y \rangle$.

Theorem (∼). A homomorphic Z is equivalent to proving the Kashiwara-Vergne statement.

Statement (∼, KV, 1978) (proven Alekseev-Meinrenken, 2006). Convolutions of invariant functions on a group matches with convolutions of invariant functions on its Lie algebra: for any finite dimentional Lie group G with Lie algebra $\mathfrak{g}$,

$$(\text{Fun}(G)^{\text{Ad}\,G}, \star) \cong (\text{Fun}(\mathfrak{g})^{\text{Ad}\,G}, \star).$$

(Closely related to the "orbit method" of representation theory).

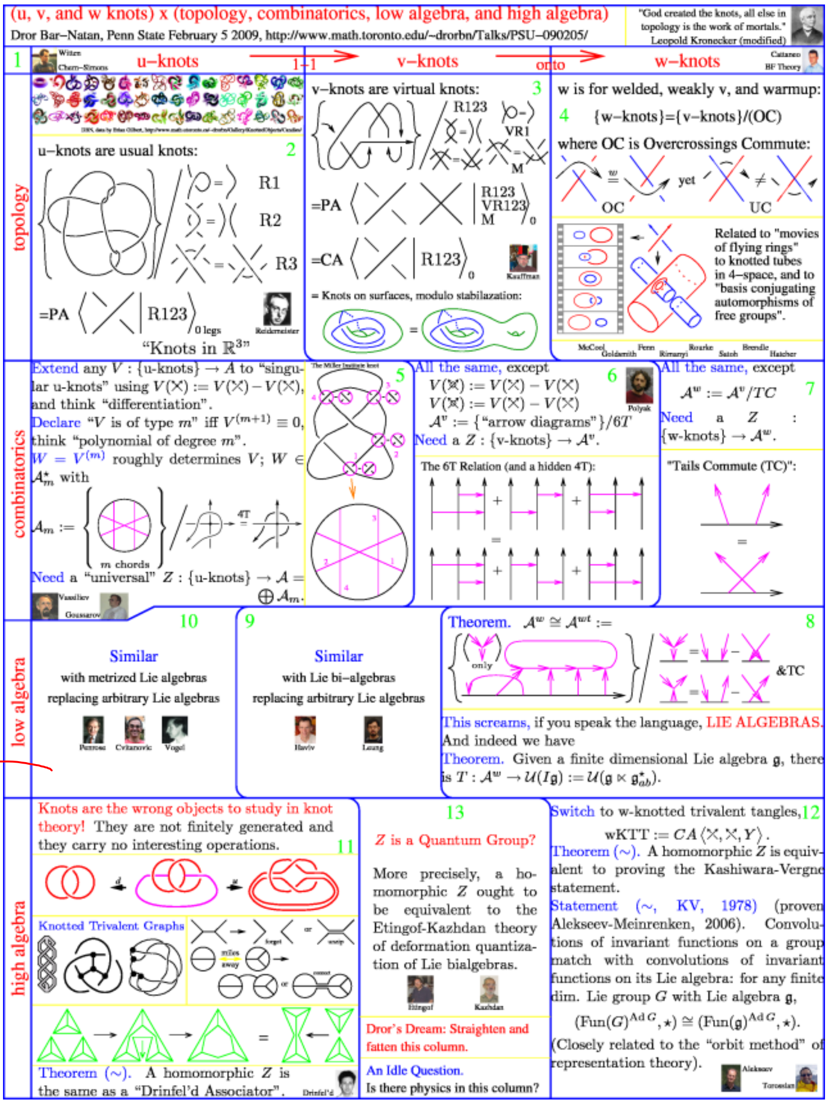
Picture of Haviv

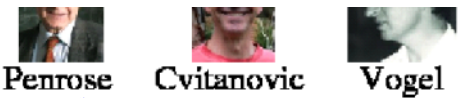
Picture of Drinfel'd.

pictures of E–K

Add: An Idle question: Is there Physics in this column?

Find a way to put pictures of Alek Tor





The Picture so far: An invariant for every $(\mathfrak{g}, R)$

$$\{u\text{-knots}\} \xrightarrow{Z} \mathcal{A} \xrightarrow[\text{given } \mathfrak{g}]{T_{\mathfrak{g}}} U(\mathfrak{g}) \xrightarrow[\text{given } R]{Tr_R} \mathbb{C}$$

high algebra. low algebra

make dashed

Dror Bar-Natan: Talks: PSU-090205: 3x4
http://www.math.toronto.edu/~drorbn/Talks/PSU-090205/3x4.html
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