Burda: Stiefel-Whitney Characteristic Classes

February-13-09
11:12 AM

Possible definitions:

1. Axiomatic definition.
2. Splitting Principle I, II.
3. $H^*(BO(n), \mathbb{Z}/2)$
4. Obstruction Theory I, II
5. Cohomology operations. (Steenrod Square)
6. An explicit construction of a Čech cocycle (McLaughlin)

X topological space (usually paracompact or even CW)

Bundles: ξ: $\begin{array}{c} E \\ \downarrow p \\ X \end{array}$ a varying family of vector spaces over a base X.
(every point has a nbd over which ξ is trivial)

Examples 1. The Möbius band.

2. The Canonical/tautological line bundle over $\mathbb{RP}^n$: $\mathcal{L}\mathbb{RP}^n \xrightarrow{p} \mathbb{RP}^n$

3. Likewise for the Grassmanian $Gr_k(\mathbb{R}^n)$:

$$\begin{array}{c} UGr_k(\mathbb{R}^n) = \{(V, x) \in Gr_k(\mathbb{R}^n) \times \mathbb{R}^n : x \in V\} \\ \downarrow \\ Gr_k(\mathbb{R}^n) \end{array}$$

4. Tangent bundles, bundles of forms, etc.

Operations 1. Pull back via $f: Y \to X$

(Example: The pull back via $z \to z^2$ of the Möbius band is the trivial) bundle

(Likewise, the pullback via $S^n \to \mathbb{RP}^n$ of the tautological bundle on $\mathbb{RP}^n$ is trivial)

2. Whitney sum: The fiber over every point is the sum of the fibers of the constituents.

Examples: a. Möb $\oplus$ Möb $=$ Trivial.

b. $TM \oplus NM$ = trivial

NM: normal bundle, M an embedded manifold.

c. $TS^n \oplus \underbrace{NS^n}_{\text{trivial}}$ = Trivial

$*$ $T\mathbb{RP}^n \oplus$ Trivial line bundle $= (\mathcal{L}\,\mathbb{RP}^n)^{\oplus n+1}$

Proof of $*$:

Axioms for Stiefel-Whitney:

1. $\xi : \begin{smallmatrix} E \\ \downarrow p \\ X \end{smallmatrix} \longmapsto w_i(\xi) \in H^i(X, \mathbb{Z}/2)$

2. $w_0(\xi) = 1 \in H^0$; $w_i(\xi) = 0$ if $i > \dim \xi$

3. Naturallity re. to pull backs.

4. The Whitney product formula:

$$w(\xi_1 \oplus \xi_2) = w(\xi_1) \cup w(\xi_2)$$

where $w = \sum w_i$

5. $w_1(\mathcal{L}\,\mathbb{RP}^1) \neq 0$

Corollaries: 1. $w(\mathcal{L}\,\mathbb{RP}^n) = 1 + (\text{a generator of } H^1(\mathbb{RP}^n)) =: 1+x$

1. Since $T\mathbb{RP}^n \oplus \text{Trivial} = (\mathcal{L}\,\mathbb{RP}^n)^{\oplus(n+1)}$,

$$w(T\mathbb{RP}^n) = (1+x)^{n+1}$$

2. If $M \to \mathbb{R}^n$ is an immersion,

$$w(TM) \cup w(NM) = 1$$

(This is an obstruction to the existence of immersions)

The splitting principle: Given $\begin{matrix} E \\ \downarrow \\ X \end{matrix}$, look for

$f: Y \to X$, s.t. a. $f^*(E) =$ a direct sum of line bundles.

b. $f^*: H^*(X) \to H^*(Y)$ is injective

(clearly if we find such an f, we can compute SW classes)

Given E, let $\mathbb{P}E \xrightarrow{\pi} X$ be the projectivization of E:

$$\begin{array}{ccc} & & E \\ & & \downarrow p \\ \mathbb{P}E & \xrightarrow{\pi} & X \end{array}$$

Claim $\pi^* E = L_{\mathbb{P}E} \oplus$ an $(n-1)$-dim bunle.

So we can continue inductively...

Need to show that π^* is injective in cohomology.