Equidecomposability:

P, Q polyhedra in $\mathbb{R}^3$;

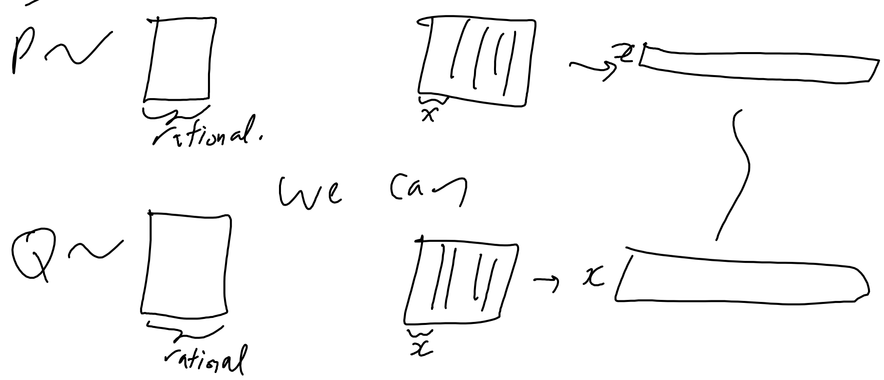
$P \sim Q$ if $P = \bigcup_{i=1}^{l} P_i$, $Q = \bigcup_{i=1}^{l} Q_i$ essentially disjoint

& P_i is congruent to Q_i

This is an equivalence relation.

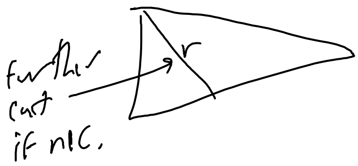
<u>Bolyai–Gerwein:</u> Any 2 polygons in $\mathbb{R}^2$ of the same area are eq-dec.

<u>Proof</u> We'll show



for if also ... we can

<u>Indeed</u> cut P into triangles each of which has a rational side



Then show that a triangle w/ a rational side can be rearranged to a rectangle,

Then use rationality to assemble all rectangles to one long one.

In 3D this is not so.

<u>Dehn Invariants</u> P: a polyhedron

$A(P) = \{\text{dihedral angles of } P\} \cup \{\pi\}$

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Given a $\mathbb{Q}$-linear $f : \langle A(P) \rangle_{\mathbb{Q}} \to \mathbb{R}$, s.t. $f(\pi) = 0$

set

$$D_f(P) = \sum_{e} l(e)\, f(\alpha(e))$$

e: edge of P $l(e)$: length of e $\alpha(e)$: dihedral angle at e.

Claim This is additive under decompositions. (easy)

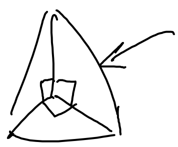
$\Rightarrow$ It is enough to find P & Q & f w/

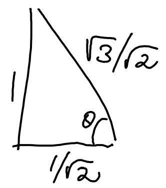
$$D_f(P) \neq D_f(Q)$$

Now $D_f(\text{Cube}) = 0$ (for any f)

$\Rightarrow$ $D_f(\text{a tetrahedron which is } 1/6 \text{ a cube}) = 0$

yet





$\theta = \arccos \frac{1}{\sqrt{3}}$