Mathcamp Quiz 2008

January-22-09
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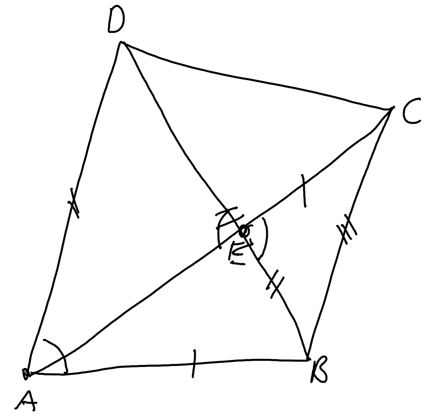
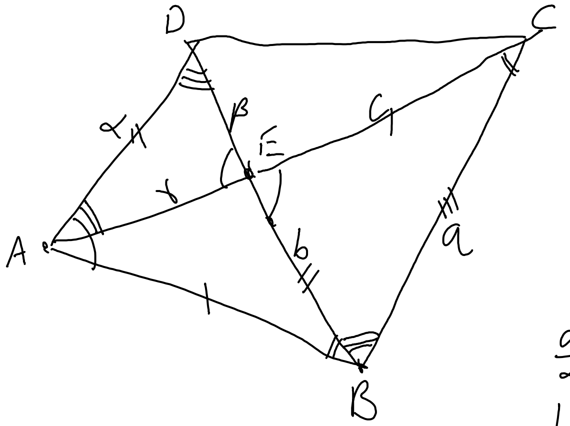
Problem 5



Find $\frac{|BC|}{|AD|} = \frac{a}{\alpha}$



$|AB| = |CE| \quad |BE| = |AD|$

$\measuredangle AED = \measuredangle BAD$

$\Rightarrow |BC| = |BD|$

$\frac{a}{\alpha} = \frac{b}{\beta} \qquad b+\beta = a$

$b = \alpha$

$\Rightarrow \frac{a}{\alpha} = \frac{\alpha}{\beta} \qquad \alpha+\beta = a$

$1+\frac{\beta}{\alpha} = \frac{a}{\alpha}$

$1+\frac{1}{x} = x$

$x^2 - x - 1 = 0$

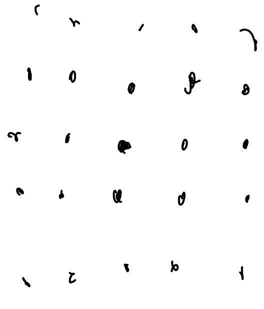
$x = \frac{1 \pm \sqrt{5}}{2}$

$x = \frac{1+\sqrt{5}}{2}$

problem 6 MCM wins

problem 7

2n steps



$\Rightarrow b_n = \sum_{k=0}^{n} \binom{2k}{k}\binom{2n-2k}{n-k}$.

$a_n = \binom{2n}{n}$

$f(x) = \sum a_n x^n$

$g(x) = \sum b_n x^n$

$g(x) = (f(x))^2$

$a = \frac{(2n)!}{\;} = \frac{2n(2n-1)(2n-2)!}{\;}$

$g(x) = (f(x))^2 \qquad a_n = \frac{(2n)!}{(n!)^2} = \frac{2n(2n-1)(2n-2)!}{n^2 (n-1)!^2}$

$$= \frac{2(2n-1)}{n} \cdot a_{n-1}$$

$$\Rightarrow \quad n a_n = (4n-2) a_{n-1} = (4(n-1)+2) a_{n-1} \quad / \cdot x^{n-1} \ \sum$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} 4(n-1) a_{n-1} x^{n-1} + 2 \sum a_{n-1} x^{n-1}$$

$$f' = \sum_{n=0}^{\infty} 4n a_n x^n + 2 \sum a_n x^n$$

$$f' = 4x f' + 2f$$

$$(1-4x)\frac{dy}{dx} = 2y$$

$$\int \frac{dy}{2y} = \int \frac{dx}{1-4x} \qquad \frac{1}{2}\log y + c = -\frac{1}{4}\log(1-4x)$$

$$2\log y = \log(1-4x) + C$$

$$y = \frac{1}{\sqrt{1-4x}}$$

$$y^2 = \frac{1}{1-4x} \qquad b_n = 4^n.$$

$$= \sum (4x)^n$$

1 →

0 1 0
1 0 1
0 1 0

→

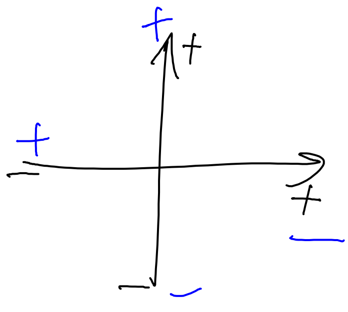
1
2 0 2
1 0 4 0 1
2 0 2
1

→

1
3 0 3
3 0 9 0 3
1 0 9 0 9 0 1
3 0 9 0 3
3 6 3
1

→ 36

$\binom{2n}{n}$: 1 2 6 20



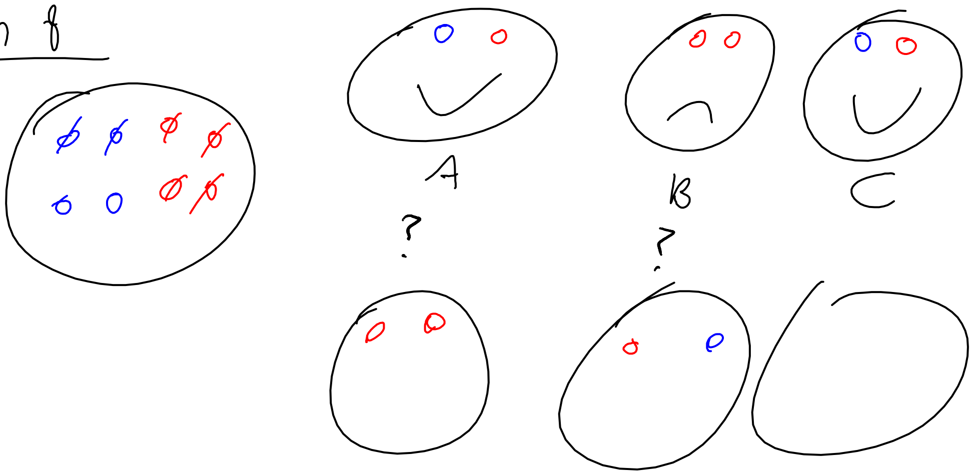
b_n 1 4 16

$$c_n = \sum_{k=0}^{n} \binom{2n}{2k}\binom{2k}{k}\binom{2n-2k}{n-k} = \sum \frac{(2n)! \cancel{(2k)!} \cancel{(2n-2k)!}}{\cancel{(2k)!(2n-2k)!} k!^2 (n-k)!^2} =$$

$$C_n = \sum_{k=0} \binom{2n}{2k}\binom{2k}{k}\binom{2n-2k}{n-k} = \sum \frac{(2n)!\,(2k)!\,\cancel{(2n-2k)!}}{\cancel{(2k)!\,(2n-2k)!}\;k!^2\,(n-k)!^2} =$$

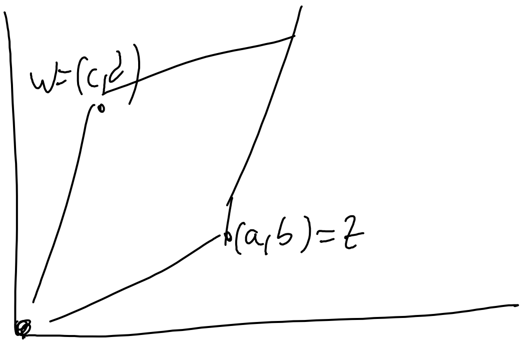
$$\sum \frac{(2n)!}{(k!(n-k)!)^2} = \sum \binom{n}{k}^2 \cdot \binom{2n}{n} = \binom{2n}{n} \sum \binom{n}{k}^2 = \binom{2n}{n}^2$$

Problem 9



Problem 10

$$\{(a,b,c,d) : (a^2+b^2)(c^2+d^2) = A^2;\; ac+bd=0\}$$



$$|z|^2|w|^2 =$$
$$= z\bar{z} \cdot w\bar{w}$$
$$= zw \cdot \overline{zw}$$
$$= |zw|^2$$

$$(a^2+b^2)(c^2+d^2) =$$
$$(ac-bd)^2 + (ad+bc)^2 = A^2$$
$$(2ac)^2 + (a+b)^2(c+d)^2 = A^2$$

$$ad + bc + ac + bd = (a+b)(c+d)$$

$$\mathrm{Re}(z/w) = 0 \qquad \frac{z}{w} = i \cdot \mathrm{const}$$

$z/w + \overline{z/w} = 0$

$$A^2 = |zw|^2 = z\bar{z}w\bar{w} = (\overset{x}{z\bar{w}})\cdot(\overset{y}{\bar{z}w}) = \overset{-x^2}{-(z\bar{w})^2}$$

$z\bar{w} + \bar{z}w = 0$

$x + y = 0$ $\qquad z\bar{w} = \pm Ai$

$$(2-i)^2(2+i)^2(1+i)^2(1-i)^2$$

$A = 4 = (1+i)^2(1-i)^2$

$A = 25 = (2+i)^2(2-i)^2$

$(2+i)^2 = 3+4i$