

we want to define a special type of categories that "look like"  $A$ -modules.

Def A category is "additive" if

\* The initial & final object coincide & are called  $0$ .

\*  $\forall X, Y$   $X \sqcup Y = X \sqcap Y$  exist & are equal.

(Exercise: it follows from the axioms that  $\sqcup$  &  $\sqcap$  are associative (in the appropriate sense))

\*  $\text{Hom}_A(X, Y)$  is always an Abelian group,

the composition of morphisms is bilinear.

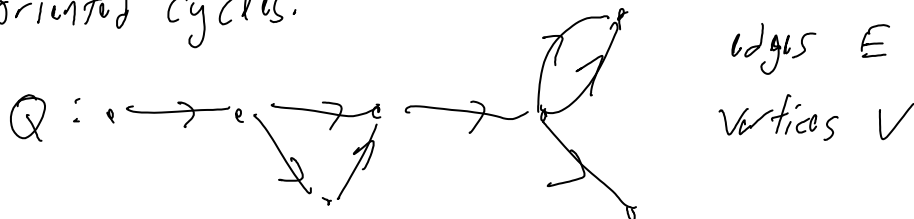
Examples: Vector spaces,  $A$ -modules,  $A$ -ring, Vector bundles on  $X$ , Topological vector spaces, Filtered vector spaces:  $(V_i \subset V_j; i < j \Rightarrow V_i \subset V_j)$

Comment In filtered vector spaces,

$$\text{Id} : (V, \{0\}) \rightarrow (V, \{V_i\})$$

is "bijective" but not invertible.

Def A "quiver" is an oriented graph without oriented cycles:



A representation of  $Q$  is a functor from the category generated by  $Q$  to  $\text{Vect}$ .

" $\text{Rep}(Q)$ " makes an additive category.

Topological Example:  $A$ : The stable homotopy category.

Topological Example:  $\mathcal{A}$ : The stable homotopy category.  
(addition of morphisms is not at all obvious)

Example: If  $\mathcal{C}$  is any category and  $\mathcal{A}$  is an additive category, then  $\text{Fun}(\mathcal{C}, \mathcal{A})$  is an additive category.

Example  $X$ : Topological space

$C^0(X)$ : objects are open subsets  
 $U \rightarrow V$  if  $U \supset V$

$\text{Fun}(C^0(X), \mathcal{A})$  is "pre-sheaves on  $X$  with values in  $\mathcal{A}$ ", aka  $\text{Pre-sh}(X)$

Example Example:  $\mathcal{O}(U) = \left\{ \begin{array}{l} \text{continuous functions} \\ \text{on } U \end{array} \right\}$

Def "Additive functor"

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Homological Algebra in an additive category

Def  $\text{Com}(\mathcal{A})$ : objects  $\cdots \rightarrow \mathcal{A} \xrightarrow{d_1} \mathcal{A} \xrightarrow{d_2} \mathcal{A} \xrightarrow{d_3} \mathcal{A} \xrightarrow{d_4} \mathcal{A} \xrightarrow{d_5} \mathcal{A} \xrightarrow{d_6} \mathcal{A} \xrightarrow{d_7} \mathcal{A} \xrightarrow{d_8} \mathcal{A} \xrightarrow{d_9} \mathcal{A} \xrightarrow{d_{10}} \mathcal{A} \xrightarrow{d_{11}} \mathcal{A} \xrightarrow{d_{12}} \mathcal{A} \xrightarrow{d_{13}} \mathcal{A} \xrightarrow{d_{14}} \mathcal{A} \xrightarrow{d_{15}} \mathcal{A} \xrightarrow{d_{16}} \mathcal{A} \xrightarrow{d_{17}} \mathcal{A} \xrightarrow{d_{18}} \mathcal{A} \xrightarrow{d_{19}} \mathcal{A} \xrightarrow{d_{20}} \mathcal{A} \xrightarrow{d_{21}} \mathcal{A} \xrightarrow{d_{22}} \mathcal{A} \xrightarrow{d_{23}} \mathcal{A} \xrightarrow{d_{24}} \mathcal{A} \xrightarrow{d_{25}} \mathcal{A} \xrightarrow{d_{26}} \mathcal{A} \xrightarrow{d_{27}} \mathcal{A} \xrightarrow{d_{28}} \mathcal{A} \xrightarrow{d_{29}} \mathcal{A} \xrightarrow{d_{30}} \mathcal{A} \xrightarrow{d_{31}} \mathcal{A} \xrightarrow{d_{32}} \mathcal{A} \xrightarrow{d_{33}} \mathcal{A} \xrightarrow{d_{34}} \mathcal{A} \xrightarrow{d_{35}} \mathcal{A} \xrightarrow{d_{36}} \mathcal{A} \xrightarrow{d_{37}} \mathcal{A} \xrightarrow{d_{38}} \mathcal{A} \xrightarrow{d_{39}} \mathcal{A} \xrightarrow{d_{40}} \mathcal{A} \xrightarrow{d_{41}} \mathcal{A} \xrightarrow{d_{42}} \mathcal{A} \xrightarrow{d_{43}} \mathcal{A} \xrightarrow{d_{44}} \mathcal{A} \xrightarrow{d_{45}} \mathcal{A} \xrightarrow{d_{46}} \mathcal{A} \xrightarrow{d_{47}} \mathcal{A} \xrightarrow{d_{48}} \mathcal{A} \xrightarrow{d_{49}} \mathcal{A} \xrightarrow{d_{50}} \mathcal{A} \xrightarrow{d_{51}} \mathcal{A} \xrightarrow{d_{52}} \mathcal{A} \xrightarrow{d_{53}} \mathcal{A} \xrightarrow{d_{54}} \mathcal{A} \xrightarrow{d_{55}} \mathcal{A} \xrightarrow{d_{56}} \mathcal{A} \xrightarrow{d_{57}} \mathcal{A} \xrightarrow{d_{58}} \mathcal{A} \xrightarrow{d_{59}} \mathcal{A} \xrightarrow{d_{60}} \mathcal{A} \xrightarrow{d_{61}} \mathcal{A} \xrightarrow{d_{62}} \mathcal{A} \xrightarrow{d_{63}} \mathcal{A} \xrightarrow{d_{64}} \mathcal{A} \xrightarrow{d_{65}} \mathcal{A} \xrightarrow{d_{66}} \mathcal{A} \xrightarrow{d_{67}} \mathcal{A} \xrightarrow{d_{68}} \mathcal{A} \xrightarrow{d_{69}} \mathcal{A} \xrightarrow{d_{70}} \mathcal{A} \xrightarrow{d_{71}} \mathcal{A} \xrightarrow{d_{72}} \mathcal{A} \xrightarrow{d_{73}} \mathcal{A} \xrightarrow{d_{74}} \mathcal{A} \xrightarrow{d_{75}} \mathcal{A} \xrightarrow{d_{76}} \mathcal{A} \xrightarrow{d_{77}} \mathcal{A} \xrightarrow{d_{78}} \mathcal{A} \xrightarrow{d_{79}} \mathcal{A} \xrightarrow{d_{80}} \mathcal{A} \xrightarrow{d_{81}} \mathcal{A} \xrightarrow{d_{82}} \mathcal{A} \xrightarrow{d_{83}} \mathcal{A} \xrightarrow{d_{84}} \mathcal{A} \xrightarrow{d_{85}} \mathcal{A} \xrightarrow{d_{86}} \mathcal{A} \xrightarrow{d_{87}} \mathcal{A} \xrightarrow{d_{88}} \mathcal{A} \xrightarrow{d_{89}} \mathcal{A} \xrightarrow{d_{90}} \mathcal{A} \xrightarrow{d_{91}} \mathcal{A} \xrightarrow{d_{92}} \mathcal{A} \xrightarrow{d_{93}} \mathcal{A} \xrightarrow{d_{94}} \mathcal{A} \xrightarrow{d_{95}} \mathcal{A} \xrightarrow{d_{96}} \mathcal{A} \xrightarrow{d_{97}} \mathcal{A} \xrightarrow{d_{98}} \mathcal{A} \xrightarrow{d_{99}} \mathcal{A} \xrightarrow{d_{100}} \mathcal{A}$   
morphisms: as expected.

$\text{Com}(\mathcal{A})$  is an additive category.

Def Homotopy between morphisms of complexes.