Blomer@GSS: What does the Borsuk-Ulam theorem have to do with quadratic forms?

January-14-09
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Def K a field

$s(K)$ = "The level of K"

$= \min_{n \in \mathbb{N}} \exists\, e_j \in K^* \text{ s.t. } \sum e_j^2 = -1$

$= \min\{n \in \mathbb{N} \mid 0 = \sum e_j^2\} - 1$

Examples $s(\mathbb{R}) = \infty$

$s(\mathbb{C}) = 1$

$s(\text{char } 2) = 1$

$q = p^n$ odd $\quad s(\mathbb{F}_q) = \begin{cases} 1 & q = 1 \bmod 4 \\ 2 & q = 3 \bmod 4 \end{cases}$

Pf $\mathbb{F}_q^* = \langle g \rangle$ $\quad$ The multiplicative group of $\mathbb{F}_q$ is cyclic

$(g^l)^2 = -1$?

$-1 = g^{\frac{q-1}{2}}$ solvable iff $2l = \frac{q-1}{2} \bmod q-1$

.....

If $q = 3 \bmod 4$, there are $\frac{q+1}{2}$ squares in $\mathbb{F}_q$, so by pigeon hole we can find two squares whose sum is 1.

Example $s(\mathbb{Q}_2) = 4$ $\quad$ ($\mathbb{Q}_2$ is 2-adic numbers)

Thm (Pfister 67) if $s(K) < \infty$ then $s(K) = 2^n$.

(Pf later)

Lemma For $n \in \mathbb{N}_0$ there is a matrix

$$T_n = T_n(x_1 \ldots x_{2^n}) \in Mat(2^n \times 2^n, K(x_1 \ldots x_{2^n}))$$

s.t. $$T_n^t T_n = \left(\sum_{j=1}^{2^n} x_j^2\right) I_{2^n}$$

Pf $T_0 = (x)$; given T_n, construct T_{n+1}:

$$T_{n+1}(\underline{\underline{x}}, \underline{\underline{y}}) = \begin{pmatrix} T_n(\underline{\underline{x}}) & T_n(\underline{\underline{y}}) \\ -T_n(\underline{\underline{y}}) & \frac{1}{\sum y_j^2} T_n(\underline{\underline{y}}) T_n^t(x) T_n(\underline{\underline{y}}) \end{pmatrix}$$

Corollary If $s(K) \geq 2^n$ then for all $a_i \in K^*$, $b_i \in K$ $i = 1 \ldots 2^n$ $\exists c_i \in K$ s.t.

$$\sum_1^{2^n} a_i^2 \sum_1^{2^n} b_i^2 = \sum_1^{2^n} c_i^2$$

Proof since $s(K) \geq 2^n$, there are never vanishing denominators in T_n, so

$$T_n = T_n(a_1 \ldots a_{2^n}) \in Mat(2^n \times 2^n, K)$$

is well defined. Let $\underline{\underline{c}} = T_n \underline{\underline{b}}$

So $$\sum a_i^2 \sum b_i^2 = \underline{\underline{b}}^t \left(\sum a_i^2\right) \underline{\underline{b}} = (T_n \underline{\underline{b}})^t (T_n \underline{\underline{b}})$$
$$= \underline{\underline{c}}^t \underline{\underline{c}} = \sum c_i^2$$

Remark with a longer proof, cor. is still correct w/o the level assumption.

Remark (Hurwitz 1898) if char $K \neq 2$,

$$\sum x_i^2 \sum y_i^2 = \sum z_i^2 \text{ with } z_i\text{'s}$$

bilinear in x_i & y_i is possible only in dim $= 1, 2, 4, 8$

Remark (Pfister 1967) if a rational function in $x_1 \ldots x_n$ is always positive (coeff's in $\mathbb{R}$) then f is a sum of at most 2^n squares.

<u>Pf of level is a power of 2</u> Assume $2^n \le s < 2^{n+1}$, so $-1 = \sum_{i=1}^{s} l_i^2$

Let $a = 1 + l_1^2 + \dots + l_{2^n-1}^2$, $b = l_{2^n} + \dots + l_s^2$

So $-1 = \frac{a}{b} = \frac{ab}{b^2} = \frac{\text{a sum of } 2^n \text{ sqrs}}{b^2}$

$= \frac{1}{b^2} \sum_1^{2^n} c_i^2 = \sum_{i=1}^{2^n} \left(\frac{c_i}{b}\right)^2$

<u>Remark</u> Every power of 2 is a level:

$$s\left(\mathbb{R}\left(x_1 \dots x_{2^n}, \sqrt{-\Sigma x_i^2}\right)\right) = 2^n$$

The level also make sense for rings!

Example $s(\mathbb{Z}/4) = 3$.

<u>Theorem</u> $\forall n \in \mathbb{Z}$ there is a ring R s.t. $s(R) = n$

<u>Pf</u> Consider pairs (X, τ) of a topological space with a continuous involution, $\tau^2 = 1$.

Make these into a category.

<u>Def</u> The level of (X, τ), $s(X, \tau)$ is the smallest integer s.t.

$$\exists f : (X, \tau) \longrightarrow (S^{n-1}, inv)$$

Given (X, τ) let

$$A_{X,\tau} := \{ f : (X, \tau) \longrightarrow (\mathbb{C}, -) \}$$

(an $\mathbb{R}$-algebra)

contravariant functor.

Theorem $s(X,\tau) = s(A_{X,\tau})$

Pf "$\geq$": assume $s(X,\tau) = n$, let $g : (X,\tau) \to (S^{n-1}, inv)$

For $j = 1, \dots, n$ let $f_j(x) = i x_j : (S^{n-1}, inv) \to (\mathbb{C}, -)$

$$\sum f(\underline{x})^2 = -1 \quad \dots \dots$$

"$\leq$"