Motivation:

$$\mathrm{Hom}_{set}(X \sqcup Y, Z) = \mathrm{Hom}_{set}(X,Z) \times \mathrm{Hom}_{set}(Y,Z)$$

$$X \sqcap Y := X \times Y$$

$$\mathrm{Hom}_{set}(Z, X \sqcap Y) = \mathrm{Hom}(Z,X) \times \mathrm{Hom}(Z,Y)$$

Category of Diagrams $\mathrm{Diag}(X \sqcup Y)$ (given some X, Y in some base category)

objects:

$$\begin{tikzcd} & X \arrow[dl] \\ Z & \\ & Y \arrow[ul] \end{tikzcd}$$

morphisms: commutative diagrams between such.

Def

$$\begin{tikzcd} X \arrow[dr] & \\ & X \sqcup Y \\ Y \arrow[ur] & \end{tikzcd}$$

the initial object in $\mathrm{Diag}(X \sqcup Y)$

Likewise, $\mathrm{Diag}(X \sqcap Y)$ has objects $\left\{ \begin{tikzcd} & X \\ Z \arrow[ur] \arrow[dr] & \\ & Y \end{tikzcd} \right\}$ with obvious morphisms

Def

$$\begin{tikzcd} & X \arrow[dl] \\ X \sqcap Y & \\ & Y \arrow[ul] \end{tikzcd}$$

~~a~~ the final object in $\mathrm{Diag}(X \sqcap Y)$

Example: Vect

Example Commutative rings with 1 Comm

$$A \sqcup B = A \otimes B \qquad A \sqcap B = A \oplus B$$

agrees with the contravariant functor

$$\mathrm{Fun}: \mathrm{Sets} \to \mathrm{Comm}$$

by $S \longmapsto \mathrm{Fun}(S) =$ functions on S

Indeed: $\mathrm{Fun}(S \times T) = \mathrm{Fun}(S) \otimes \mathrm{Fun}(T)$

Indeed, $\mathrm{Fun}(S\times T) = \mathrm{Fun}(S)\otimes \mathrm{Fun}(T)$

$\mathrm{Fun}(S\sqcup T) = \mathrm{Fun}(S)\oplus \mathrm{Fun}(T)$

Example C = Rings (not nec. commutative)

$A \sqcap B = A\oplus B$

$A \sqcup B =$ the free product of A & B

Example C = Groups (not nec. Abelian)

$G\sqcap H = G\times H$ (direct product)

$G\sqcup H = G * H$ (free product)

Motivation C = A-modules

$f: M \longrightarrow N$

$$\begin{tikzcd} \ker(f) \arrow[r, "k"] \arrow[d] & M \arrow[d, "f"] \\ 0 \arrow[r] & N \end{tikzcd}$$

$\mathrm{Diag}(\ker(f))$:= The category with objects

$$\begin{tikzcd} X \arrow[r] \arrow[d] & M \arrow[d, "f"] \\ 0 \arrow[r] & N \end{tikzcd}$$

(obvious morphisms)

$\ker(f)$ is a final object in $\mathrm{Diag}(\ker(f))$

Def Let $\mathcal{C}$ be a category with an object 0 which is both initial & final. Given $f: X \longrightarrow Y$ in $\mathcal{C}$, $\ker(f)$ is a final object in the likewise-defined $\mathrm{Diag}(\ker(f))$

Example For the category of pointed sets,

$\ker(f) = f^{-1}(*)$

Motivation $f: M \to N$

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d] & N \arrow[d] \\ 0 \arrow[r] & \operatorname{coker} f \end{tikzcd}$$

$\operatorname{Diag}(\operatorname{coker}(f)) =$

$$\left\{ \begin{tikzcd} M \arrow[r, "f"] \arrow[d] & N \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd} \right\}$$

obvious morphisms

Def Cokernels in general are the initial objects in $\operatorname{Diag}(\operatorname{Coker}(f))$

Example In pointed sets, $\operatorname{coker}(f) = Y/f(x) \sim *$

Example $\mathcal{C}$ = Top Vector spaces; there are morphisms w/ vanishing ker & coker, yet which aren't iso:

$$V^{\text{discrete}} \xrightarrow[id]{} V$$