Meinrenken

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10:09 AM

W v.s. w/ quadratic form $B(\cdot,\cdot)$

$W \subset Cl(W)$ the Clifford algebra:

$$\langle W : v_1v_2 + v_2v_1 = B(v_1,v_2)\rangle$$

Eg., given V, $W := V \oplus V^*$

Suppose B is split (i.e., $W \cong \mathbb{R}^{n,n}$)

or $\exists\, L \subset V$ with $L^\perp = L$

Q: Is there a proper meaning to all this? Will get us $\Lambda(V)$?

A "spinor module" for $Cl(W)$ is an irred. module.

Given $L \subset W$ ($\in Lag(W)$) as above, set $S = Cl(W)/Cl(W)L$

is a spinor module. Any two are isomorphic

w/ $Hom_{cl}(S,S')$ is $\mathbb{R}^*$

In the $W = V \oplus V^*$,

$$S = Cl(V)/Cl(V)\cdot V \cong \Lambda(V^*)$$

$\rho(v,\alpha)\varphi = \iota(v)\varphi + \alpha\wedge\varphi$

If $\varphi = 1$, $N_\varphi = V$

$\varphi \in \Lambda^{top}(V^*)\setminus\{0\} \Rightarrow N_\varphi = V^*$

$\varphi = e^w$ if $w \in \Lambda^2(V^*) \Rightarrow$

$$\rho(v,\alpha)e^w = (\iota(v)w + \alpha)e^w = 0$$

$$\Leftrightarrow \iota(v)w + \alpha = 0$$

So $N_\varphi = \{(v,\alpha) : \iota(v)w + \alpha = 0\}$

Q: Is the symplectic version of this relating to E-K at a Weyl algebra?

Q What remains of W-facts in the Weyl algebra case?

Most general, take $Q \subset V$, $\mu \in \Lambda^{top}((V/Q)^*)$,

$w \in \Lambda^2 Q^*$ and set $\varphi = e^w \cdot \mu$

Suppose S is any spinor module, $\rho : Cl(W) \to End(S)$

For any $\varphi \in S$ non-zero,

If $W = V \oplus V^*$

$$N_\varphi = \{ w \in W \mid \rho(w)\varphi = 0 \}$$

Then N_φ is isotropic: $N_\varphi \subset N_\varphi^\perp$

<u>Def</u> φ is "pure" if N_φ is Lagrangian

So we have $\{\text{pure spinors}\} \longrightarrow \{\text{Lagrangian spaces}\}$

This is onto! If L is Lag, $\exists$ pure φ ....

we have

$$\mathbb{R}^* \longrightarrow \{\text{pure spinors}\} \longrightarrow Lag(W)$$

There is a bilinear pairing on pure spinors:

If $W = V \oplus V^*$, $S = \Lambda(V^*)$

$$(\varphi, \psi) = (\varphi^T \wedge \psi)_{top} \in \Lambda^{top}(V^*)$$

If $\varphi = v_1 \wedge \ldots \wedge v_k$, $\varphi^T = v_k \wedge \ldots \wedge v_1$

$$(\varphi, \rho(x)\psi) = \pm(\rho(x^T)\varphi, \psi)$$

<u>Thm</u> (Cartan) If φ & ψ are pure spinors then

$$N_\varphi \cap N_\psi = 0 \iff (\varphi, \psi) \neq 0.$$

<u>Eg</u> $\varphi = 1$, $\psi = \mu \in \Lambda^{top}(V^*)$

$N_\varphi = V$, $N_\psi = V^*$

$V \cap V^* = 0$ $(\varphi, \psi) = \mu \neq 0$.

$\{W = V \oplus V^*\}$ form a category with objects W's

Morphisms $=$ $Lag(W' \times W^-)$

$\mathrm{Mor}(W, W') \ni \Lambda$

write

$$x \sim_\Lambda x' \Leftrightarrow (x, x') \in \Lambda$$

Inner products descend to the quotient by $\sim_\Lambda$

Another category $\mathrm{Obj} = \{(W, L)\}$

with morphisms that carry L to L'

in $\mathrm{mor}((W, L), (W', L'))$

A subcategory: $\{(W, V^*)\}$

morphisms $\Phi : V \to V'$ along with $w \in \Lambda^2 V^*$

$$\Lambda = \{(v, \alpha, v', \alpha') \in W \times \overline{W'} : v' = \phi(v), \alpha = \Phi^* \alpha' + \iota_v w\}$$

Now consider M = manifold, $\mathbb{T}M := TM \oplus T^*M$

$$Cl(\mathbb{T}M) \circlearrowright S = \Lambda T^* M$$

$$\varphi \in \Gamma(\Lambda T^* M) = \Omega(M) \quad \text{"pure spinors"}$$

$N_\varphi \subset \mathbb{T}M$ a Lagrangian subbundle.

Let $\eta \in \Omega^3(M)$, $d\eta = 0$

Shlomo: No symplectic analog.

<u>Def</u> φ is η-integrable if

$$(d + \eta)\varphi = \rho(w)\varphi \quad \text{for some } w \in \Gamma(\mathbb{T}M) \text{ ?} \; (Cl(\mathbb{T}M))$$

on $\Gamma(\mathbb{T}M)$ define a bracket:

"Courant bracket" student of Weinstein

not a Lie bracket! $[\![w_1, w_2]\!]$ by

$$\rho([\![w_1, w_2]\!]) = [[d + \eta, \rho(w_1)], \rho(w_2)]$$

Let $E \subset \mathbb{T}M$ be a lagrangian subbundle

E is "integrable" if $\Gamma(E)$ is closed under $[\![\,,\,]\!]$

Fact: Then $[\,,\,]_E = [\![\,,\,]\!]|_{\Gamma(E)}$ is a Lie bracket

Fact φ is integrable $\Leftrightarrow$ N_φ is integrable

Example $M = G$ a Lie group, $\mathfrak{g} = \mathrm{Lie}(G)$ with a metric

$$E = \mathrm{span}\{ e(\xi) \mid \xi \in \mathfrak{g} \} \subset \mathbb{T}G = TG \oplus T^*G$$

where

$$e(\xi) = \left(\xi^L - \xi^R, \frac{\theta^L + \theta^R}{2} \cdot \xi \right)$$

left & right vector field extensions of ξ $\quad \theta^L = g^{-1} dg \qquad \theta^R = dg \cdot g^{-1}$

integrable rel. $\quad \eta = \frac{1}{12} \theta^L \cdot [\theta^L, \theta^L] \in \Omega^3(G)$

E is integrable $\quad [\![e(\xi_1), e(\xi_2)]\!] = e([\xi_1, \xi_2])$

(book with cohomology of Lie groups
Greub - Halperin - Vanstone)