CMO 2008

July-28-08
8:07 PM

40th Canadian Mathematical Olympiad

Wednesday, March 26, 2008

---

1. $ABCD$ is a convex quadrilateral for which AB is the longest side. Points M and N are located on sides AB and BC respectively, so that each of the segments AN and CM divides the quadrilateral into two parts of equal area. Prove that the segment MN bisects the diagonal BD.

2. Determine all functions f defined on the set of rational numbers that take rational values for which

$$f(2f(x) + f(y)) = 2x + y ,$$

for each x and y.

3. Let a, b, c be positive real numbers for which $a + b + c = 1$. Prove that

$$\frac{a - bc}{a + bc} + \frac{b - ca}{b + ca} + \frac{c - ab}{c + ab} \le \frac{3}{2} .$$

4. Determine all functions f defined on the natural numbers that take values among the natural numbers for which

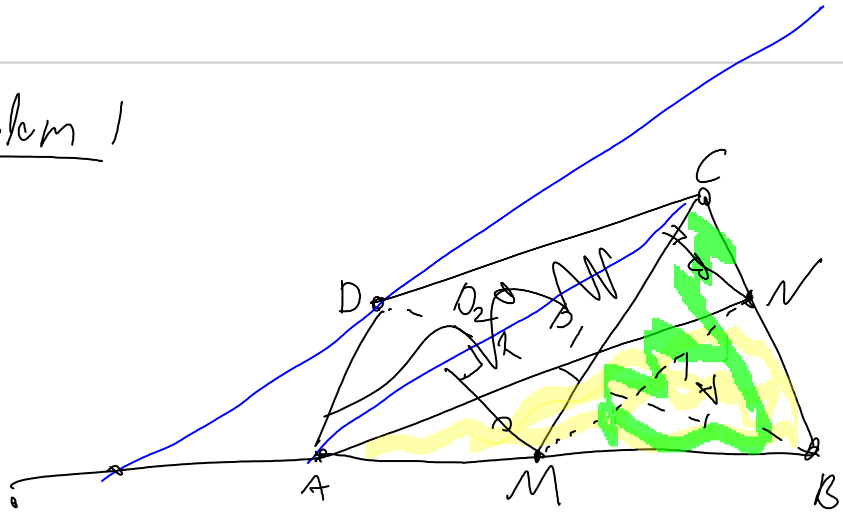
$$(f(n))^p \equiv n \mod f(p)$$

for all $n \in \mathbf{N}$ and all prime numbers p.

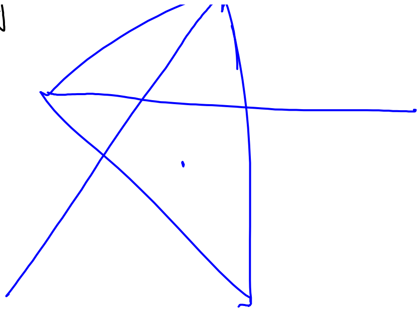
5. A *self-avoiding rook walk* on a chessboard (a rectangular grid of unit squares) is a path traced by a sequence of moves parallel to an edge of the board from one unit square to another, such that each begins where the previous move ended and such that no move ever crosses a square that has previously been crossed, *i.e.*, the rook's path is non-self-intersecting.

Let $R(m, n)$ be the number of self-avoiding rook walks on an $m \times n$ (m rows, n columns) chessboard which begin at the lower-left corner and end at the upper-left corner. For example, $R(m, 1) = 1$ for all natural numbers m; $R(2, 2) = 2$; $R(3, 2) = 4$; $R(3, 3) = 11$. Find a formula for $R(3, n)$ for each natural number n.

Problem 1



$B + D_1 = CD_1$



Problem 2 Determine

$$\{f:\mathbb{Q}\to\mathbb{Q} : \forall x,y \ \ f(2f(x)+f(y)) = 2x+y\}$$

$x=y=0$ $f(3f(0))=0$ $\qquad a=f(0)$ $\qquad f(3a)=0$

$x=3a$ $y=0$ $\qquad f(f(0)) = 6a$ $\qquad f(a)=6a$

$x=0$ $y=3a$ $\qquad f(2f(0))=3a$

$\qquad f(2a)=3a$

$x=2a$ $y=0$ $\qquad f(7a)=4a$

$x=a$ $y=0$ $\qquad f(13a)=2a$

$x=0$ $y=2a$ $\qquad f(5a)=2a$

$x=0$ $y=5a$ $\qquad f(5a)=10a$

$x=2a$ $y=3a$ $\qquad f(6a)=7a$

$x=a$ $y=3a$ $\qquad f(12a)=5a$

x	$f(x)$
0	a
a	$6a$
$2a$	$3a$
$3a$	0
$4a$	$10a$
$5a$	$2a \neq 10a$
$6a$	$7a$
$7a$	$4a$
$8a$	$17a$
$9a$	
$10a$	
$11a$	
$12a$	$5a$

$x=y \quad \Rightarrow \quad f(3f(x)) = 3x$

$y=0 \Rightarrow f(2f(x)+a) = 2x$

$x=0 \Rightarrow f(2a+f(y))=y$

Say $f(u)=x$, $f(v)=y$

$\Rightarrow$

Assuming $f(0)=0$, put $x=0$:

$$f(f(y))=y$$

$f(x+y)$

Thus

$$2f(x)+f(y)=f(2x+y)$$

$y=0$ $\quad 2f(x)=f(2x)$

$\Rightarrow 2f(\frac{x}{2})=f(x)$

$x=y$ $\quad 3f(x)=f(3x)$

$x \mapsto x/2$

$$f(x)+f(y)=f(x+y)$$

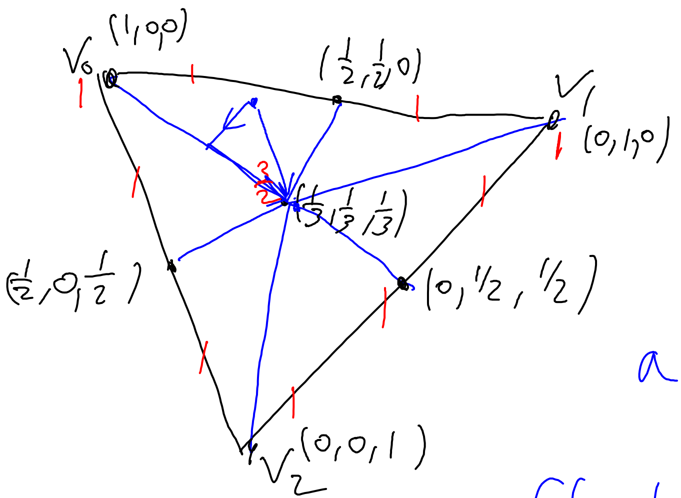
$\Rightarrow$ If $f(1)=b$ Then

$$f(n)=nb$$

$$\Rightarrow f\left(\frac{n}{m}\right)=\frac{n}{m}b$$

$$\Rightarrow f(x)=bx \Rightarrow f(x)=\pm x$$

Problem 3 "Barrycentric Coordinates

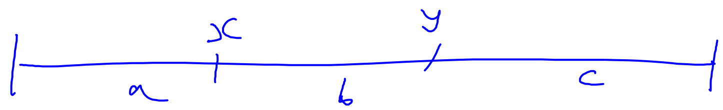


$a_i \geq 0$

$$\left\{\sum a_i v_i : \sum a_i = 1\right\}$$

$a \geq b \geq c$

$f(a,b,c)$



$a=x \qquad b=y-x \qquad c=1-y$

$$\underline{x-(y-x)(1-y)} \quad , \quad \underline{(y-x)-x(1-y)}$$

$$\frac{x-(y-x)(1-y)}{x+(y-x)(1-y)} + \frac{(y-x)-x(1-y)}{}$$

$$\left(a, \frac{b}{2}, \frac{b}{2}\right)$$

Problem 4

$n^p = n \mod p.$

Lemma $\quad (a+b)^p = a^p + b^p \mod p.$

$$\| \quad a^p + \underbrace{pa^{p-1}b + \binom{p}{2}a^{p-2}b^2 + \dots}_{0} + b^p$$

$$\binom{p}{k} = \frac{p!}{k!(p-k)!}$$

Now $n^p = n \mod$ by induction on n.

$f(n)^2 = n \mod f(2)$

$f(n)^p = p \mod f(p)$ $\qquad p \mid f(p)$

~~$f(p)^p = p$ mod~~

$0 \mod f(p) = f(p)^p = p \mod f(p)$

$f(n)^p \mod xp = m \qquad f(n)^p \mod p = n$

$f(n)^p \mod p = f(n)$

$\forall_p^{\text{prime}} \quad p \mid (f(n) - n)$

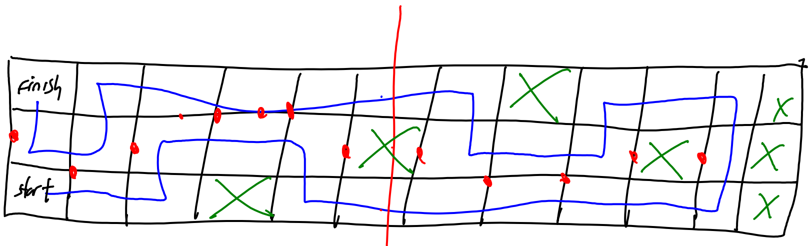
$f(n)^p = n \mod f(p)$
therefore $\mod p$

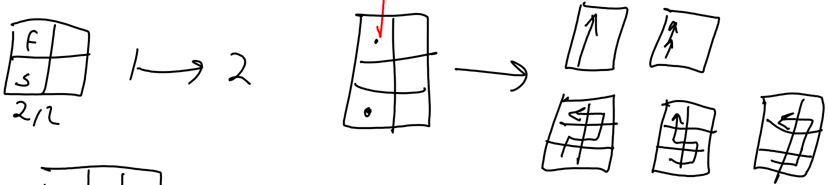
$f(n)^p = f(n) = n \mod p$

$\Rightarrow f(n) = n \quad \bmod \text{ every } p$

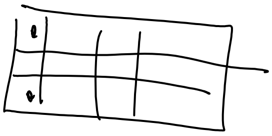
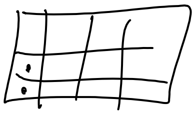
$\Rightarrow f(n) = n,$

Problem 5





$R(2,n) = n$

$F(n) =$  $g(n) =$ 

$F(n+1) = 1 +$

	29	12	5	2	1
	41	17	7	3	1
	29	12	5	2	1

$$a_{n+1} = a_n + b_n \qquad \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$$
$$b_{n+1} = 2a_n + b_n$$

$S = 2 \quad P = -1$

$$\lambda^2 - 2\lambda - 1 = 0$$
$$\lambda_{1,2} = \frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}$$