Brav@Cooolstuff: Morita equivalence

November-28-08
3:17 PM

From "Morita theory in abelian, derived and stable model categories"
Authors: Stefan Schwede

Pasted from <http://arxiv.org/abs/math/0310146>

Moral: "what matters is how you relate to others"

Defn Two gadgets of the same kind are Morita equivalent if their categories of modules are equivalent.

Example If two rings are Morita equivalent (have same Mod-R) then their centres are isomorphic.

Claim $Z(R) \cong End(Id) = \left\{ \begin{array}{l} \text{natural transformations} \\ \text{from } Id \text{ to } Id \end{array} \right\}$

where $Id: \text{Mod-}R \longrightarrow \text{Mod-}R$ is the identity functor.

Example $S = M_{n\times n}(\mathbb{C})$, $R = \mathbb{C}$ are M.E.

Suppose $\text{Mod-}R \xrightarrow[\sim]{F} \text{Mod-}S$

what's special about $S \in \text{Mod-}S$?

a. S is "small": $\oplus Hom(S, M_i) = Hom(S, \oplus M_i)$.

b. S is projective

$Hom(S, -)$ is exact; $M \longrightarrow N \longrightarrow 0$, $S \to N$, $\exists\, S \dashrightarrow M$

c. S "generates" Mod-S — given M, $\exists I$ & a surjection $S^I \longrightarrow M \longrightarrow 0$

$\Rightarrow$ $\exists P \in \text{Mod-}R$ s.t. $F(P) \cong S$. P must

$\Rightarrow$ $\exists P \in$ Mod-R s.t. $F(P) \cong S$. P must have all those same properties. Also note

$$\mathrm{End}_R(P) \cong \mathrm{End}_S(S) \cong S$$

suggests $F = \mathrm{Hom}_R(P, -)$

<u>Thm</u> (Morita, 1958) TFAE

1. R & S are Morita equivalent.
2. $\exists$ small, projective generator for $P \in$ Mod-R s.t. $\mathrm{End}_R(P) = S$, which is an S-R bimodule.
3. The functor
$$\text{Mod-}R \xrightarrow{\mathrm{Hom}_R(P, -)} \text{Mod-}S$$
is an equivalence of categories.

Back to the example $R = \mathbb{C}$, $S = M_n(R)$.

Take $P = R^n$. So

$$\begin{aligned} \text{Mod-}R &\xrightarrow{\sim} \text{Mod-}S \\ M &\longrightarrow \mathrm{Hom}_R(R^n, M) \\ R &\longrightarrow \mathrm{Hom}_R(R^n, R) = \text{"row vectors"} \end{aligned}$$