Bergeron: Combinatorial hopf algebras and towers of algebras

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* Intro (Hopf Algebras, Sym, REP S_n)
* How to generalize.
* A rigidity Thm of Zelevinsky
* Thm (LLB)
* Idea of proof [Fomin; stanley].

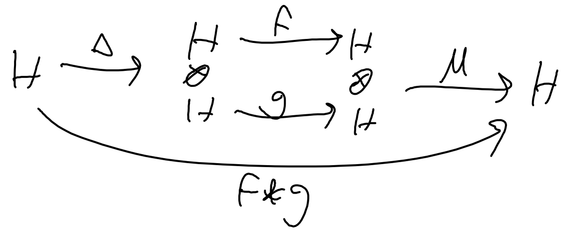
H: Free $\mathbb{Z}$-module

$\mu : H \otimes H \to H$, unit $u : \mathbb{Z} \to H$, some axioms.
$\Delta : H \to H \otimes H$, $\epsilon : H \to \mathbb{Z}$, some axioms.

Some compatability axiom.

$S : H \to H$ an antipode w/ some axioms.

$F, g : H \to H$ linear map. (e.g. $u \circ \epsilon$, I)



the convolution.

S is the inverse of I relative to $*$

There is a notion of a "graded" Hopf algebra.
"Connected" means $H_0 = \mathbb{Z}$.
(In the connected case, Bialg $\Rightarrow$ Hopf)

$SYM_{(m)} := \mathbb{Z}[x_1 \ldots x_m]^{S_m} \Leftarrow$ invariants.

$\cong \mathbb{Z}[e_1 \ldots e_n]$ where $(e_0 = 1)$

$$\sum e_k t^k = \prod_{i=1}^{m} (1+x_i t)$$

$$\sim \quad \mathbb{Z}[h_1 \dots h_m]$$

$$1 = \prod_i^m (1-x_i t) \underbrace{\prod_{i=1}^m \frac{1}{1-x_i t}}_{\sum h_i t^i}$$

$$SYM := \varprojlim SYM_{(m)} = \mathbb{Z}[e_1 \dots] = \mathbb{Z}[h_1 \dots]$$

$\deg e_k = k \qquad \deg h_k = k$

$\Delta : SYM \longrightarrow SYM \otimes SYM$ via

$$h_k \longmapsto \sum_{i=0}^{k} h_i \otimes h_{k-i}$$

Rep of S_n : $\mathbb{C}S_n$ semi-simple

Finitely many irreps of S_n, indexed by conjugacy classes $\Leftrightarrow$ partitions $(\lambda_1 \dots \lambda_\ell)$ "$\lambda \vdash n$"

$\lambda_1 \geq \lambda_2 \geq \dots \lambda_\ell > 0, \ \sum \lambda_i = n$

$SYM = \bigoplus SYM_n \leftarrow$ Hom. comp. of deg n.

$\dim SYM_n = \#\{\lambda \vdash n\}$

$K_0(\mathbb{C}S_n) = \mathbb{Z}$-module spanned by iso. classes of irreps of S_n

$\bigoplus_{n \geq 0} K_0(\mathbb{C}S_n) \longleftrightarrow SYM$ (unique Hopf map) up to the single Aut. of SYM

$\uparrow$

Hopf structure: mult: $K_0(\mathbb{C}S_n) \otimes K_0(\mathbb{C}S_m) \longrightarrow K_0(\mathbb{C}S_{n+m})$

$\mu \qquad N \otimes M \quad \Uparrow \quad N \otimes M$

induced rep'n.

comult by restriction of rep'n.

Δ

$$\tilde{M} \mapsto \sum \mathrm{RES}^{S_n}_{S_i \times S_{n-i}} M$$

The compatability of μ & Δ is "Mackey's formula".

M^λ (irreps) $\longrightarrow$ S_λ Schur functions, a linear basis of SYM

So SYM is a Hopf algebra with a priveledged basis.

$$S_\lambda S_\mu = \sum C^\nu_{\lambda\mu} S_\nu, \quad C^\nu_{\lambda\mu} \geq 0$$

$$\Delta(S_\nu) = \sum C^\nu_{\lambda\mu} S_\lambda \otimes S_\mu$$

So $\{S_\lambda\}$ is a self dual basis.

So (SYM, S_λ) is connected, graded, positive (i.e., $C^\nu_{\lambda\mu} \geq 0$), self-dual

<u>Thm</u> (Zelevinsky) Any H with all those properties is $SYM^{\otimes r}$.

<u>Thm</u> Removing self-duality, (H, H^*) coming from a $K_0(A) \cong \oplus K_0(A_n)$ - - - -

$$\Rightarrow \dim A_n = r^n n!$$