# Friedgut@GB244: Exposing Juntas and Dictatorships

September-12-08
11:12 AM

## Arrow's theorem (1951)

$n$ voters, 3 candidates $a, b, c$

Social Welfare Function:

$$\text{SWF: } F : \underbrace{\left(S_{\{a,b,c\}}\right)^n}_{\text{each voter orders the candidates}} \to S_{\{a,b,c\}}$$

Such that:

The original Arrow thm has a weaker neutrality requirement

1. $F$ commutes with permutations ("$F$ is neutral"):

$$F(\sigma) = F\begin{pmatrix}\sigma_1 \\ \vdots \\ \sigma_n\end{pmatrix} \qquad F\begin{pmatrix}\alpha\sigma_1 \\ \vdots \\ \alpha\sigma_n\end{pmatrix} = \alpha F(\sigma)$$

2. Indifference to Independent Alternatives: (IIA)

$$a \overset{?}{>}_{F(\sigma)} b \text{ depends only on } \begin{pmatrix} a >_{\sigma_1} b \\ \vdots \\ a >_{\sigma_n} b \end{pmatrix}$$

Thm The only SWF are dictatorships and anti-dictatorships.

(majority won't work:

$$\left.\begin{array}{ll} 1/3 & a>b>c \\ 1/3 & b>c>a \\ 1/3 & c>a>b \end{array}\right\} \text{2/3 prefer } a \text{ to } b,\ b \text{ to } c,\ c \text{ to } a$$

---

## Erdös Ko Rado 1960 (proven 1938) $r < \frac{n}{2}$

$$A \subset \binom{[n]}{r} \text{ intersecting: } F, G \in A \Rightarrow F \cap G \neq \emptyset$$

$\max|A| =$ pick one element and all others containit $= \binom{n-1}{r-1}$,

[illegible]

$\max |\mathcal{A}| =$ pick one element and all others must contain it $= \binom{n-1}{r-1}$,

and the unique extremal examples are "dictatorships" as above.

---

<u>Dinur, Friedgut:</u> All intersecting families are essentially contained in a junta:

$r, n, \mathcal{A}$ as before. Then

$\exists \epsilon\ \exists \mathcal{J} \subseteq \binom{[n]}{r}$ s.t.

1. $|\mathcal{A} \setminus \mathcal{J}| \leq \epsilon \binom{n-1}{r-1}$
2. $\mathcal{J}$ depends on $\text{few}(\epsilon, \frac{r}{n})$ coordinates
3. $\mathcal{J}$ is "almost" intersecting

not defined here

and if $o(n)$ then $\left[\binom{n}{r} \gg \binom{n-1}{r-1} \gg \binom{n-2}{r-2}\right]$

$r/n$ ratios $r/n$

$\forall$ intersecting $\mathcal{A}$ $\exists i$ (the dictator) s.t.

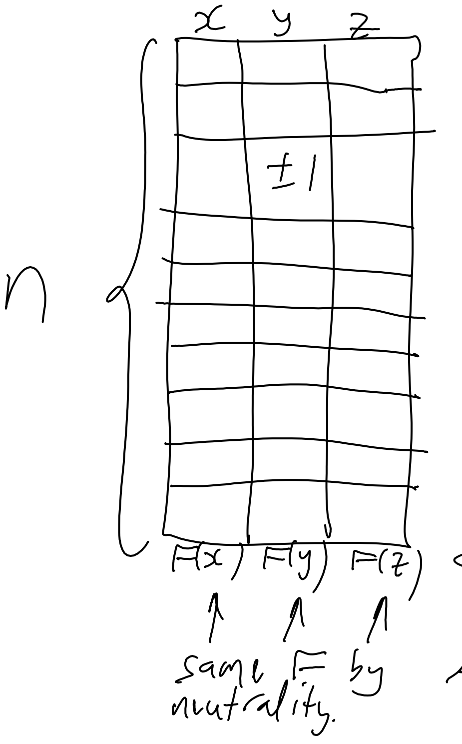
$$|\{F \in \mathcal{A} : i \notin F\}| \leq \binom{n-2}{r-2}$$

---

Proof of Arrow due to Gil Kalai:

If $F$ is a neutral, IIA SWF which is acyclic on $(1-\epsilon)$ of the inputs then it (dis)agrees with a certain voter $1-10\epsilon$ of the time.

$a>b$? $a>c$? $b>c$?




$6^n$ possible inputs, as the inputs must be consitant

Take uniform probabilities

may be cyclic!

Also, $F(-x) = -F(x)$

$$E(x_i) = 0 \quad E(x_i^2) = 1 \quad E(x_i x_j) = 0 \quad i \neq j$$

$$E\Big(\prod_{i \in I} x_i\Big) = \begin{cases} 0 & I \neq \emptyset \\ 1 & I = \emptyset \end{cases}$$

$$E\Big(\Big(\prod_I x_i\Big)\Big(\prod_J x_j\Big)\Big) = \delta_{I,J}$$

So $\{\prod_I x_i\}_I$ is a complete O.N system on $\{\pm 1\}^n$. So any $F$ has an expansion

$$F = \sum_I a_I \prod_{i \in I} x_i \quad \text{so} \quad E(F) = a_\emptyset$$

(as $F$ is $\{\pm 1\}$-valued) $\quad 1 = E(F^2) = \sum a_I^2$

$$E(x_i y_i) = 1/3$$

by enumerating $S_{\{a,b,c\}}$

$$E\Big(\Big(\prod_I x_i\Big)\Big(\prod_J y_j\Big)\Big) = \left(\frac{1}{3}\right)^{|I|} \delta_{I,J}$$

$$P(F \text{ is acyclic}) = 3P\begin{pmatrix} a >_F b \\ a >_F c \end{pmatrix} = \frac{3}{2} P(\text{"a" wins or loses "2"})$$

$$= \frac{3}{2} P(F(x) = F(y))$$

$$= \frac{3}{4}(1 + E(F(x)F(y)))$$

Aside

$E(F(x)F(y)) =$
$P(F(x) = F(y))$
$- P(F(x) \neq F(y))$
yet sum is 1

So

$$P(F \text{ is acyclic}) = 1 \iff E(F(x)F(y)) = 1/3$$

"almost" $\iff$ "almost"

Yet

$$E(F(x)F(y)) = \sum_{|I| \geq 1} a_I^2 \left(\frac{1}{3}\right)^{|I|}$$

$$\leq \frac{1}{3} \sum a_I^2 = \frac{1}{3}$$

equality iff $a_{|I|} = 0$ for $|I| > 1$

So $F$ is linear!

$F = \sum a_i X_i$ but as $F$ is $(\pm 1)$-valued,

$F = x_j$ for some $j$. !!!

For stability one only needs to also know that if $F$ is almost linear and boolean then $F$ is almost a dictatorship.

comments: 1. proof of $P(X=Y) = \frac{1+E(XY)}{2}$ was ugly

2. Proof of "linear $\Rightarrow$ dictatorship"

was ugly.

3. A conceptual ideology was missing!