Fixing J

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Claim There exists some function J so that whenever D is a derivation on a Lie algebra $\mathfrak{g}$, we have for all $y \in \mathfrak{g}$,

$$e^{-y} D(e^{y}) = J(\operatorname{ad} y) D y$$

What's J? Take $D = \operatorname{ad} z$:

$$e^{-y} D(e^{y}) = e^{-y}(z e^{y} - e^{y} z) = e^{-y} z e^{y} - z$$

$$= e^{-\operatorname{ad} y} z - z = \left(\frac{e^{-\operatorname{ad} y} - 1}{\operatorname{ad} y} \operatorname{ad} y\right)(z)$$

$$= \frac{e^{-\operatorname{ad} y} - 1}{\operatorname{ad} y}(-\operatorname{ad} z(y)) = \frac{1 - e^{-\operatorname{ad} y}}{\operatorname{ad} y}(\operatorname{ad} z(y))$$

$$\Rightarrow J(x) = \frac{1 - e^{-x}}{x}$$

$$z e^{y} = e^{y} e^{-\operatorname{ad} y} z$$

$$yz = (\operatorname{ad} y) z + z y$$

$$e^{y} z = e^{\operatorname{ad} y}(z) e^{y}$$

$$z e^{-y} = e^{-y} e^{\operatorname{ad} y}(z)$$

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Claim $e^{A + \epsilon B} \overset{\epsilon^2}{=} e^{A}\left(1 + \epsilon \frac{1 - e^{-\operatorname{ad} A}}{\operatorname{ad} A} B\right)$

Pf

$$\operatorname{ad} A\left(e^{A + \epsilon B}\right) \overset{?}{=} e^{A}\left(\epsilon\left(1 - e^{-\operatorname{ad} A} B\right)\right)$$

$$\operatorname{ad} A\left(e^{A + \epsilon B}\right) \overset{?}{=} \epsilon\left(e^{A} B - B e^{A}\right)$$

✓ by telescopic summation

Works ignoring terms in Ker ad A