D, exp, log

August-12-08
8:20 AM

$$\frac{d}{dt}\exp\sum t^k a_k = \left(\sum k t^{k-1} a_k\right)\exp\sum t^k a_k$$

$\Longrightarrow$ multiply by Z^{-1} $\left(Z = \exp\sum t^k a_k\right)$, set $t=1$, and get $\sum k a_k$

wrong! see below.

$BCH(x,y) := \log e^x e^y = \sum a_k$ where a_k is a Lie poly of degree k.

$BCH(tx,ty) = \log e^{tx}e^{ty} = \sum a_k t^k$

Thus $\sum k t^{k-1} a_k = e^{-ty}e^{-tx}\frac{d}{dt} e^{tx}e^{ty}$

(taking above $Z = \exp BCH = e^{tx}e^{ty}$)

$$= e^{-ty}\cancel{e^{-tx}}\cancel{e^{tx}}x e^{ty} + e^{-ty}e^{-tx}e^{tx}e^{ty}y$$

$$= e^{-ty}xe^{ty} + y \sim e^{-t\,\mathrm{ad}\,y}x + y$$

Too good to be true!

$$\frac{-e^{-x}+1}{x} = 1 - \frac{x}{2} + \frac{x^2}{6}$$

$$\overset{!!}{J(x)} \qquad \frac{1-(1-x+\frac{x^2}{2}-\frac{x^3}{6})}{x}$$

$$\frac{d}{dt}e^{A(t)} = e^{A(t)}\frac{1-e^{-\mathrm{ad}\,A(t)}}{\mathrm{ad}\,A(t)}\frac{d}{dt}A(t)$$

follows from

$$e^{A+\epsilon B} = e^A\left(1+\epsilon\frac{1-e^{-\mathrm{ad}A}}{\mathrm{ad}A}B\right),$$

proven below

$[B, A^k] =$

$$(A+\epsilon B)^n = A^n + \epsilon\sum_{k=1}^n A^{k-1}BA^{n-k}$$

$=$

Get approx

$$\left(1+\frac{\mathrm{ad}\,BCH}{2}+\frac{\mathrm{ad}^2 BCH}{6}\cdots\right)\left(\sum ka_k\right) = \frac{-e^{-\mathrm{ad}\,BCH(x,y)}+1}{\mathrm{ad}\,BCH(x,y)}\left(\sum ka_k\right) = e^{\mathrm{ad}y}x + y$$

Let $E\left(\sum a_k\right) := \sum k a_k$. Our equation is of the general form

$$J(\mathrm{ad}\,\phi)(E\phi) = L$$

our unknown. a given Lie polynomial

<u>Claim</u> $\emptyset$ exists and is unique, and it is a Lie polynomial.

It would be worthwhile to reformulate this as a general technique for graded Lie groups / Lie algebras!

$$\left[\sum a_k, \sum k a_k\right] = \sum_{i,j}[a_i, j a_j] = \sum_{i<j}[a_i, j a_j] + [a_j, i a_i]$$
$$= \sum_{i<j}(j-i)[a_i, a_j]$$

$$BA = AB - (\mathrm{ad}\,A)(B)$$

$$e^{A+\epsilon B} = e^A\left(1+\epsilon\frac{1-e^{-\mathrm{ad}\,A}}{\mathrm{ad}\,A}B\right) + O(\epsilon^2)$$

$$\left(\frac{1-e^{-\mathrm{ad}A}}{\mathrm{ad}A}\right)(B) = B - \frac{[A,B]}{2} + \frac{[A,[A,B]]}{6} \cdots$$

check:

$$1 + A + \epsilon B + \frac{(A+\epsilon B)^2}{2} + (A+\epsilon B)^3/6 = 1 + A + \frac{A^2}{2} + \frac{A^3}{6} + \epsilon\left(B + \frac{AB+BA}{2} + \frac{AAB+ABA+BAA}{6}\right)$$

$$= 1 + A + \frac{A^2}{2} + \frac{A^3}{6} + \epsilon\left(B + AB - \frac{1}{2}[A,B] + \frac{A^2}{2}B - \frac{1}{6}A[A,B] - \frac{1}{3}A[A,B] + \frac{1}{6}[A,[A,B]]\right)$$

checks.

<u>Proof</u> A co-product decides participation/non-participation, and of those that participate, one is a B.