

```

Int[K_, col_] := K /. {
  Pos[a_, b_, c_, d_] =>
    H[{a, b}, -1] + H[{a, c}, -Colour[a, col] + 1] +
    H[{a, d}, Colour[b, col]] + H[{b, c}, -1] + H[{c, d}, -Colour[b, col]],

  Neg[a_, b_, c_, d_] =>
    H[{a, b}, -Colour[b, col]] + H[{a, d}, Colour[b, col]] +
    H[{b, c}, -1] + H[{b, d}, 1 - Colour[a, col]] + H[{c, d}, -1],

  P[a_, b_] => H[{a}, 1] + H[{b}, -1],

  CD[a_, c_, b_, d_] => H[{a, b}, (1/2) * (Colour[a, col] + Colour[b, col])] +
    H[{a, c}, -Colour[b, col]] + H[{a, d}, (1/2) * (Colour[a, col] - Colour[b, col])] +
    H[{b, c}, (1/2) * (Colour[a, col] - Colour[b, col])] + H[{b, d}, Colour[a, col]] +
    H[{c, d}, (1/2) * (-Colour[a, col] - Colour[b, col])],

  Arr[a_, b_, c_, d_] =>
    H[{b, c}, -Colour[a, col]] + H[{b, d}, Colour[a, col]] + H[{c, d}, -Colour[b, col]],

  HBlob[a_, b_] => H[{a}, Colour[a, col]] + H[{b}, -Colour[a, col]] }

H /: H[l_List, a_] ** H[k_List, b_] :=
  H[Sort[Join[l, k]], Signature[Ordering[Join[l, k]]] * a * b]
H /: H[l_List, a_] + H[l_, b_] := H[l, a + b]
H /: k_ * H[l_List, a_] := H[l, k * a]
H[l_List, a_] /; Length[Union[l]] < Length[l] := 0
H /: 0 ** H[k_List, b_] := 0
H /: H[k_List, b_] ** 0 := 0
H /: H[l_List, 0] := 0

Colour[i_, col_] := t[Position[col, i][[1, 1]]]

Prod[{a_}] := a
Prod[l_List] := l[[1]] ** Prod[Drop[l, 1]]

Proj[L_, n_] :=
  L /. H[l_List, a_] /; Take[l, n] != Range[n] -> 0 /. H[l_List, a_] => H[Drop[l, n], a]

Sig[K_, n_, m_] :=
  Signature[Ordering[Select[List @@ Join @@ Apply[List, K, {1}], m >= # > n &]]]

AT[K_, n_, m_, col_] :=
  Proj[Sig[K, n, m] * Distribute[Prod[Int[K, col]], n] /. H[l_List, a_] => H[l, Expand[a]]]

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"Test for Chord Diagrams - Relation"
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```
K[1] = {CD[4, 2, 3, 1], CD[2, 5, 1, 6]}
```

```
K[2] = {CD[2, 5, 3, 1], CD[4, 2, 1, 6]}
```

```
K[3] = {P[3, 1], P[4, 2], P[2, 5], P[1, 6]}
```

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Test for Chord Diagrams - Relation
```

```
{CD[4, 2, 3, 1], CD[2, 5, 1, 6]}
```

```
{CD[2, 5, 3, 1], CD[4, 2, 1, 6]}
```

```
{P[3, 1], P[4, 2], P[2, 5], P[1, 6]}
```

```
AT[K[1], 2, 4, {{2, 4, 5}, {1, 3, 6}}] - AT[K[2], 2, 4, {{2, 4, 5}, {1, 3, 6}}] ===  
Expand[(t[1] * t[2]) * AT[K[3], 2, 4, {{2, 4, 5}, {1, 3, 6}}]]
```

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True
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