Groups and Lie algebras

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From Papadima-Suciu's arxiv:math/0307087

1.1. A classical construction of W. Magnus associates to a group G a graded Lie algebra over $\mathbb{Z}$,

$$\mathrm{gr}(G) = \bigoplus_{k\geq 1} \Gamma_k G/\Gamma_{k+1} G,$$

where $\{\Gamma_k G\}_{k\geq 1}$ is the lower central series of the group, defined inductively by $\Gamma_1 G = G$ and $\Gamma_{k+1} G = (\Gamma_k G, G)$, and the Lie bracket $[x, y]$ is induced from the group commutator $(x, y) = xyx^{-1}y^{-1}$.

Many properties of a group are reflected in properties of its associated graded Lie algebra. For instance, if G is finitely generated, then the abelian groups $\mathrm{gr}_k(G)$ are also finitely generated; their ranks, $\phi_k(G)$, are important numerical invariants of G. In the case when $G = \mathbf{F}_n$, the free group of rank n, Magnus showed that $\mathrm{gr}(\mathbf{F}_n) = \mathbf{L}_n$, the free Lie algebra on n generators, whose graded ranks were computed by E. Witt. In general, though, the computation of the LCS ranks $\phi_k(G)$ can be exceedingly difficult.

Question How are the Lie algebras

1. $gr(G)$ (Magnus)
2. $Proj(quandle(G))$
3. M_G, the Malcev completion of G
4. $h(G,\mathbb{Q})$, the rational holonomy Lie algebra of G

related to each other for a general group G?

At least the first two are so simply defined that they ought to be very simply related.

Question Is

$$proj(G) \simeq \mathcal{U}(gr(G))$$