Ober handout corrections

May-19-08
2:19 PM

Projectivization, Welded ~~Knots~~ Tangles and Alekseev–Torossian

The Categorification Speculative Paradigm.

- Every object in math is the Euler characteristic of a complex.
- Every operation in math lifts to an operation between complexes.
- Every identity in mathematics is true up to homotopy.

The Projectivization Tentative Speculative Paradigm. Projectivization?

- Every graded algebraic structure in mathematics is the projectivization of a plain ("global") one.
- Every equation written in a graded algebraic structure is an equation for a homomorphic expansion, or for an automorphism of such.

Graded Equations Examples

- $e(x+y)=e(x)e(y)$ in $\mathbb{Q}[[x,y]]$.
- The pentagon and hexagons in $\mathcal{A}(\uparrow_{3,4})$.
- The equations defining a QUEA, the work of Etingof and Kazhdan.
- The Alekseev-Torossian equations in $\mathcal{U}(\mathrm{sder}_n)$ and $\mathcal{U}(\mathrm{tder}_n)$.

sder $\leftrightarrow$ tree-level $\mathcal{A}$
tder $\leftrightarrow$ more

$$F\in\mathcal{U}(\mathrm{tder}_2);\quad F^{-1}e(x+y)F=e(x)e(y)\quad\Longleftrightarrow\quad F\in\mathrm{Sol}_0$$

$$\Phi=\Phi_F:=(F^{12,3})^{-1}(F^{1,2})^{-1}F^{23}F^{1,23}\in\mathcal{U}(\mathrm{sder}_3)$$

$$\Phi^{1,2,3}\Phi^{1,23,4}\Phi^{2,3,4}=\Phi^{12,3,4}\Phi^{1,2,34}\qquad\text{"the pentagon"}$$

$$t=\frac{1}{2}(y,x)\in\mathrm{sder}_2\text{ satisfies }4T\quad\text{and}\quad r=(y,0)\in\mathrm{tder}_2\text{ satisfies }6T$$

$$R:=e(r)\text{ satisfies Yang-Baxter: }R^{12}R^{13}R^{23}=R^{23}R^{13}R^{12}$$

$$\text{also }R^{12,3}=R^{13}R^{23}\text{ and }F^{23}R^{1,23}(F^{23})^{-1}=R^{12}R^{13}$$

$$\tau(F):=RF^{21}e(-t)\text{ is an involution},\Phi_{\tau(F)}=(\Phi_F^{321})^{-1}$$

$$\mathrm{Sol}_0^\tau:=\{F\colon\tau(F)=F\}\text{ is non-empty; for }F\in\mathrm{Sol}_0^\tau,$$


Alekseev

$$e(t^{13}+t^{23})=\Phi^{213}e(t^{13})(\Phi^{231})^{-1}e(t^{23})\Phi^{321}$$

$$\text{and}\quad e(t^{12}+t^{13})=(\Phi^{132})^{-1}e(t^{13})\Phi^{312}e(t^{12})\Phi$$

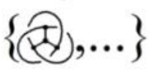

Torossian

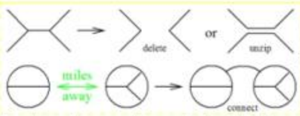
This is just a part of the Alekseev–Torossian work!

So What?

- Related to the Kashiwara-Vergne Conjecture!
- Will likely lead to an explicit tree-level associator, a linear equation away from a 1-loop equation, two linear equations away from a 2-loop associator, etc.!
- A baby version of the QUEA equations; we may be on the right tracks!

Knotted Trivalent Graphs

$\mathcal{O}(\mathbb{A})=$ 

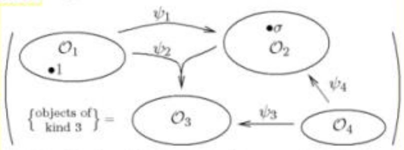


Theorem. KTG is generated by the unknotted $\mathbb{A}$ and the Möbius band, with identifiable relations between them.
Theorem. $Z(\mathbb{A})$ is equivalent to an associator Φ.

Algebraic Knot Theory

Theorem. {ribbon knots} $\sim\{u\gamma\colon\gamma\in\mathcal{O}(\circ\!\circ),\ d\gamma=\bigcirc\!\bigcirc\}$.
Hence an expansion for KTG may tell us about ribbon knots, knots of genus 5, boundary links, etc.

"An Algebraic Structure"

$\mathcal{O}=$ 

- Has kinds, objects, operations, and maybe constants.
- Perhaps subject to some axioms.
- We always allow formal linear combinations.

Defining proj $\mathcal{O}$. The augmentation "ideal":

$$I=I_{\mathcal{O}}:=\left\{\begin{matrix}\text{formal differences of ob-}\\\text{jects "of the same kind"}\end{matrix}\right\}.$$

Then $I^n:=\left\{\begin{matrix}\text{all outputs of algebraic}\\\text{expressions at least } n \text{ of}\\\text{whose inputs are in } I\end{matrix}\right\}$, and

$$\mathrm{proj}\,\mathcal{O}:=\bigoplus_{n\geq 0}I^n/I^{n+1}.$$

- Has same kinds and operations, but different objects and axioms.

Knot Theory Anchors.

- $(\mathcal{O}/I^{n+1})^\star$ is "type n invariants".
- $(I^n/I^{n+1})^\star$ is "weight systems".
- proj $\mathcal{O}$ is $\mathcal{A}$, "chord diagrams".


Vassiliev
Goussarov

Warmup Examples.

- The projectivization of a group is a graded associative algebra.
- A quandle: a set Q with a binary op $\wedge$ s.t.

$1\wedge x=1,\quad x\wedge 1=x\wedge x=x,$ (appetizers)
$(x\wedge y)\wedge z=(x\wedge z)\wedge(y\wedge z).$ (main)

$L:=\mathrm{proj}\,Q$ is a graded Lie algebra: set $\bar{v}:=(v-1)$ (these generate I!), feed $1+\bar{x}$, $1+\bar{y}$, $1+\bar{z}$ in (main), and collect the surviving terms of lowest degree:

$$(\bar{x}\wedge\bar{y})\wedge\bar{z}=(\bar{x}\wedge\bar{z})\wedge\bar{y}+\bar{x}\wedge(\bar{y}\wedge\bar{z}).$$

An Expansion is $Z\colon\mathcal{O}\to\mathrm{proj}\,\mathcal{O}$ s.t. $Z(I^n)\subset(\mathrm{proj}\,\mathcal{O})_{\geq n}$ and $Z_{I^n/I^{n+1}}=Id_{I^n/I^{n+1}}$ (A "universal finite type invariant"). In practice, it is hard to determine proj $\mathcal{O}$, but easy to guess a surjection $\rho:\mathcal{A}\to\mathrm{proj}\,\mathcal{O}$. So find $Z'\colon\mathcal{O}\to\mathcal{A}$ with $Z'(I^n)\subset\mathcal{A}_{\geq n}$ and $Z'_{I^n/I^{n+1}}\circ\rho_n=Id_{\mathcal{A}_n}$:

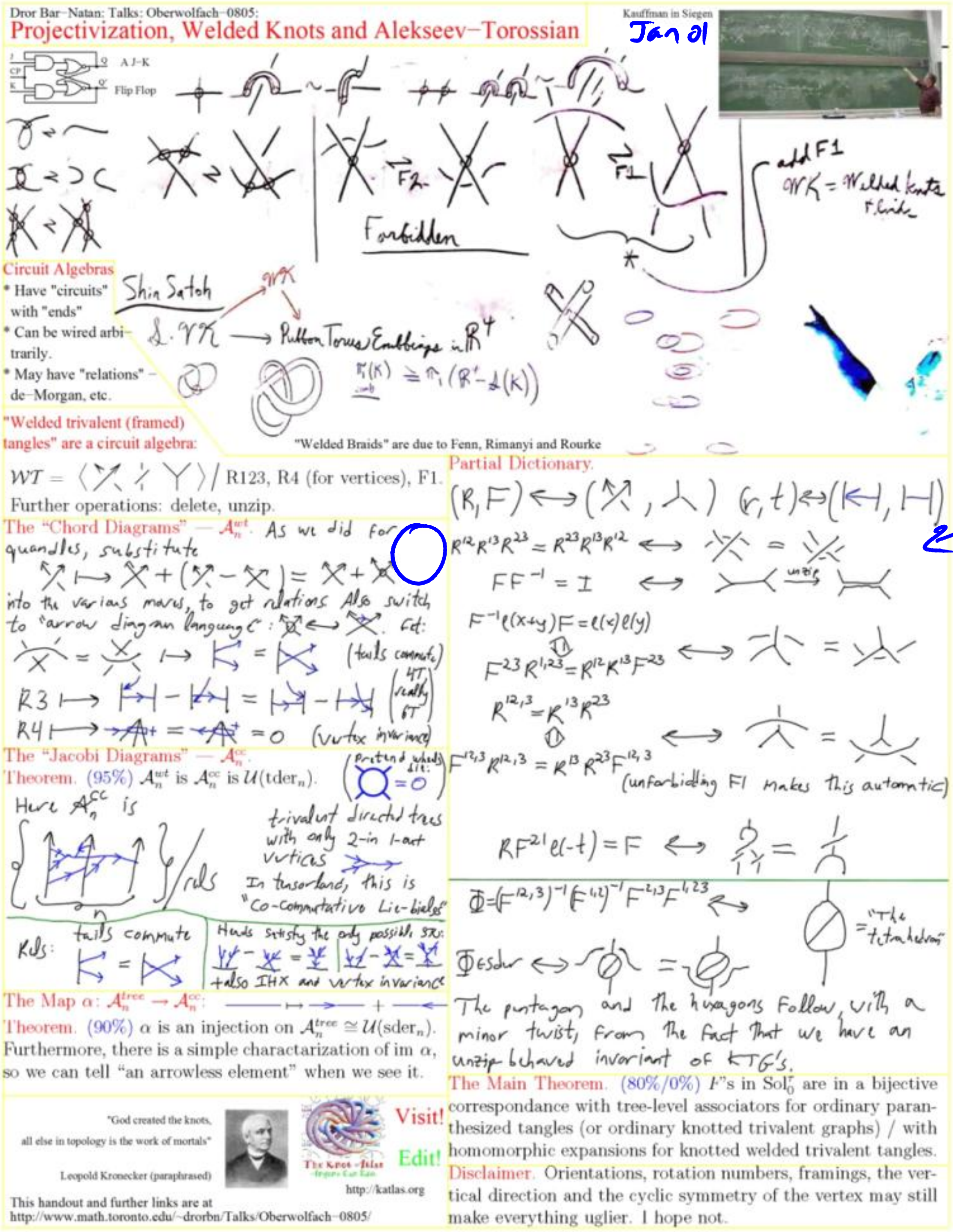
$$\begin{tikzcd}
\mathcal{O} \arrow[r, "Z'"] \arrow[dr, "Z"'] & \mathcal{A} \arrow[r, leftrightarrow] \arrow[d, "\rho"] & \mathcal{A}_n \\
 & \mathrm{proj}\,\mathcal{O} \arrow[r, leftrightarrow] & (\mathrm{proj}\,\mathcal{O})_n \arrow[u, "Z'_{I^n/I^{n+1}}"']
\end{tikzcd}$$

Can you make this diagram less confusing?

Homomorphic Expansions are expansions that intertwine the algebraic structure on $\mathcal{O}$ and proj $\mathcal{O}$. They provide finite / combinatorial handles on global problems.


X–S. Lin

A word on f.g. $\mathcal{O}$'s.



Dror Bar-Natan: Talks: Oberwolfach-0805:

Projectivization, Welded Knots and Alekseev-Torossian

Kauffman in Siegen

Jan 01

A J-K Flip Flop

Forbidden

add F1

wK = Welded knots

Circuit Algebras

* Have "circuits" with "ends"
* Can be wired arbitrarily.
* May have "relations" – de-Morgan, etc.

Shin Satoh

wK → Ribbon Torus Embeddings in $\mathbb{R}^4$

$\pi_1(K) \cong \pi_1(\mathbb{R}^4 - \delta(K))$

"Welded trivalent (framed) tangles" are a circuit algebra:

"Welded Braids" are due to Fenn, Rimanyi and Rourke

$wT = \langle \ \rangle /$ R123, R4 (for vertices), F1.

Further operations: delete, unzip.

The "Chord Diagrams" — $\mathcal{A}_n^{wt}$

As we did for quandles, substitute into the various moves, to get relations. Also switch to "arrow diagram language":

R3 ⟼

R4 ⟼ $= 0$ (vertex invariance)

(tails commute) (4T, really 6T)

The "Jacobi Diagrams" — $\mathcal{A}_n^{cc}$

Theorem. (95%) $\mathcal{A}_n^{wt}$ is $\mathcal{A}_n^{cc}$ is $\mathcal{U}(\mathrm{tder}_n)$.

(Pretend wheels die: $= 0$)

Here $\mathcal{A}_n^{cc}$ is

$/$ rels

trivalent directed trees with only 2-in 1-out vertices

In tensorland, this is "Co-Commutative Lie-bialg"

Rels: tails commute

Heads satisfy the only possible STU: + also IHX and vertex invariance

The Map α: $\mathcal{A}_n^{tree} \to \mathcal{A}_n^{cc}$:

Theorem. (90%) α is an injection on $\mathcal{A}_n^{tree} \cong \mathcal{U}(\mathrm{sder}_n)$. Furthermore, there is a simple characterization of im α, so we can tell "an arrowless element" when we see it.

"God created the knots, all else in topology is the work of mortals"

Leopold Kronecker (paraphrased)

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http://katlas.org

This handout and further links are at http://www.math.toronto.edu/~drorbn/Talks/Oberwolfach-0805/

Partial Dictionary

$(R, F) \longleftrightarrow (\ ,\ )$ $(r, t) \longleftrightarrow (\ ,\ )$

$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12} \longleftrightarrow$

$FF^{-1} = I \longleftrightarrow$ unzip

$F^{-1}e(x+y)F = e(x)e(y)$

$F^{23}R^{1,23} = R^{12}R^{13}F^{23} \longleftrightarrow$

$R^{12,3} = R^{13}R^{23}$

$F^{12,3}R^{12,3} = R^{13}R^{23}F^{12,3}$ (unforbidding F1 makes this automatic)

$RF^{21}e(-t) = F \longleftrightarrow$

$\Phi = (F^{12,3})^{-1}(F^{1,2})^{-1}F^{2,3}F^{1,23} \longleftrightarrow$ = "The tetrahedron"

Φ esdr $\longleftrightarrow$

The pentagon and the hexagons follow, with a minor twist, from the fact that we have an unzip-behaved invariant of KTG's.

The Main Theorem. (80%/0%) F's in Sol_0^r are in a bijective correspondance with tree-level associators for ordinary parenthesized tangles (or ordinary knotted trivalent graphs) / with homomorphic expansions for knotted welded trivalent tangles.

Disclaimer. Orientations, rotation numbers, framings, the vertical direction and the cyclic symmetry of the vertex may still make everything uglier. I hope not.

licts of Polyak & Viro.