ON ACROBAT TOWERS AND QUANTIZATION OF POISSON STRUCTURES ON LINEAR SPACES

DROR BAR-NATAN

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Abstract. Very abstract.

Contents

1. Introduction

This space is reserved for the introduction.

1.1. **Acknowledgement.** No paper is complete withoout one.

2. Acrobats and acrobat towers

An acrobat is a two-legged creature that also has a head. The two legs of an acrobat are distinct. One is called "the left leg" and the other is called "the right leg". When acrobats are drawn in the plane with their heads up, we normally do not specify which leg is left and which is right. Thus my $(5-\epsilon)$ years old son Assaf would not be able to understand this paper; readers that can tell left from right should have no difficulty.

Let X be a finite set. An X-based acrobat tower (or simply "a tower", if X is clear from the context), is a structure T made of $|X|$ bases labeled by the elements of X, and a finite number of acrobats (called "the degree" of the tower) placed on top of these bases in an iterative manner; the first acrobat is placed with his legs on two of the bases, and each subsequent acrobat is placed with his legs on either one of the bases or on a head of a

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previously placed acrobat. The union of the set X of bases with the set of heads of acrobats in T is the set of "footholds" of T. See figure 1 for an example.

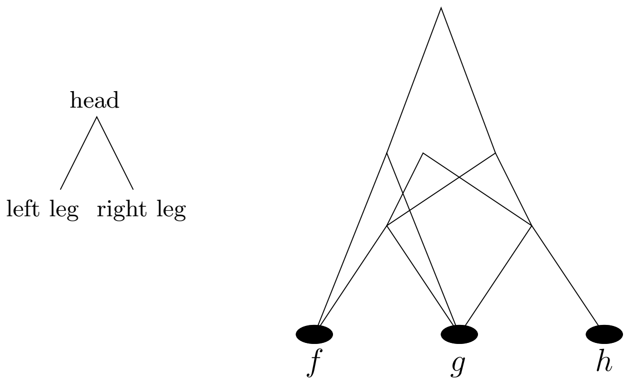
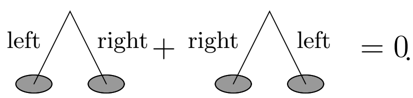


Figure 1. An acrobat and an $\{f, g, h\}$-based acrobat tower. Each vertex (except the bases) is the head of an acrobat. The two edges going down from any given verttex are the legs of that acrobat. The edges going up from any vertex (head) are the legs of the acrobats standing on that head. There are 6 heads in this picture, so it represents a degree 6 acrobat tower. There are $9 = 3 + 6$ footholds on this tower.

Definition 2.1. Let $\mathcal{T}(X)$ be the graded linear space freely generated by all X-based acrobat towers, modulo the (homogenuous) "antisymmetry" and "Jacobi" relations. The antisymmetry relation states that if the two legs of any given acrobat in a tower are flipped, the tower reverses its sign:



The Jacobi relation is a bit harder to state. A degree m "Jacobi relation tower" is defined in the same way as an acrobat tower, only that it is made of $m - 2$ standard acrobats like the one in figure 1, and one special acrobat (the "Jacobi" acrobat) that has three legs instead of just two, numbered 1, 2, and 3. In the simplest case, the Jacobi acrobat has nothing on its head. In this case, the Jacobi relation corresponding to the Jacobi relation tower is obtained by replacing the Jacobi acrobat by a pair of standard "daughter" acrobats in three different ways, adding the corresponding acrobat towers, and setting the sum to be 0. An example is in figure 2. In the general case, there's also some number k of legs lying on the head of the Jacobi acrobat. In this case, to get the corresponding Jacobi relation one also has to sum over the 2^k possible ways of dividing those k legs between the two heads of the two daughter acrobats (so the relation involves a total of $3 \cdot 2^k$ towers). An example is in figure 3.

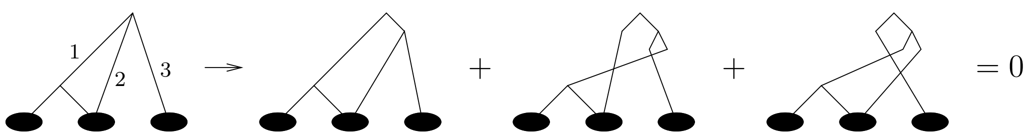


Figure 2. A simple Jacobi relation tower, and the corresponding Jacobi relation.

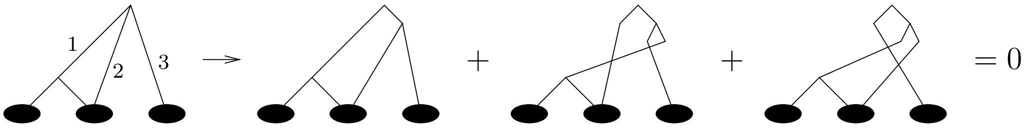


Figure 3. A general Jacobi relation tower (with $k = 2$), and the corresponding Jacobi relation. Only the immediate vicinity of the Jacobi acrobat is shown.

3. The realization of an acrobat tower

4. Quantization towers and quatization

4.1. **Some operations on towers.** Let us now describe a few simple operations that involve acrobat towers. The simplest operation is the operation of "base substitution", and there's not much to say about it. If T is a tower in (say) $\mathcal{T}(\{1,2\})$, we will denote by $T(f,g)$ (say), the tower obtained from T by renaming the bases of T in the obvious way ($1 \to f$, $2 \to g$). A more complicated operation is the operation of "tower composition":

Definition 4.1. Let T_1 be a $\{1,2\}$-based tower and let T_2 be an $\{3,4\}$ based tower. (the choice of the base sets is arbitrary, they don't even have to be of equal sizes. We make this simple choice only for the purpose of illustration). The composition $T_1(f, T_2(g,h))$ a sum of $\{f,g,h\}$-based towers constructed as follows: (see example in figure 4)

(1) Remove base 2 of T_1, and replace it by a (small-scale) copy of T_2, summing over all possible ways of attaching the legs of T_1 that used to rest on base 2 to the footholds (bases and heads) of T_2. If T_1 has l legs on base 2 and T_2 has k footholds, this is a sum of k^l towers.
(2) Perform the base substitutions $1 \to f$, $3 \to g$, and $4 \to h$ on all towers in the sum. (Base 2 no longer exists so we don't need to specify how to rename it).

The operation of composition can be extended linearily to an operation on formal sums of towers, and as such it respects the antisymmetry and the Jacobi relations. Therefore it makes sense as an operation $\mathcal{T}(\{1,2\}) \otimes \mathcal{T}(\{3,4\}) \to \mathcal{T}(\{f,g,h\})$. It is clear how to define similar substitution operations such as $T_1(T_2(f,g),h)$ and more complicated substitution operations such as $T_1(T_2(f,g), T_3(T_4(h,j)))$.

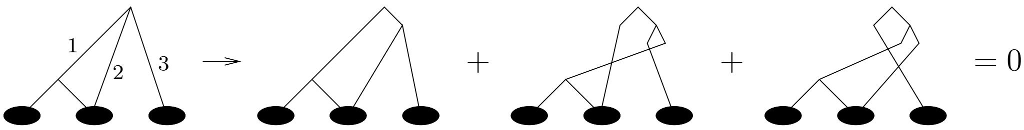
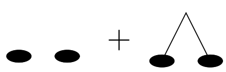


Figure 4. A substitution operation

Definition 4.2. A quantization tower is an element $\star$ of $\mathcal{T}(\{1,2\})$ satisfying:

(1) Normalization:

$\star =$  + (higher degree towers).

(2) Associativity:

$$\star(\star(f,g),h) = \star(f, \star(g,h)).$$

Using infix notation for $\star$, this is simply $(f \star g) \star h = f \star (g \star h)$.

We can finally state our main theorem:

Theorem 1. *Quantization towers exist.*

5. A cohomological interlude

6. The existance of quantization towers

References

[Ko] Kontsevich's paper.

Institute of Mathematics, The Hebrew University, Giv'at-Ram, Jerusalem 91904, Israel
E-mail address: `drorbn@math.huji.ac.il`